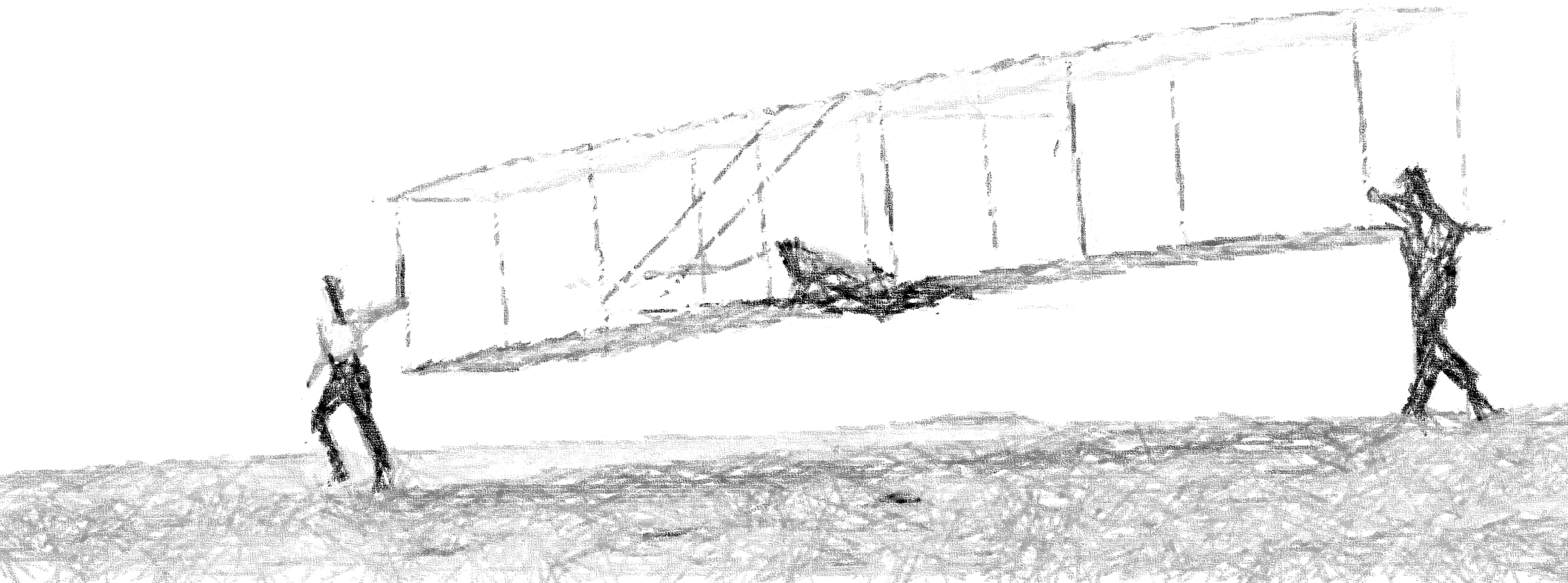
uc3m

Mechanics applied to Aerospace Engineering

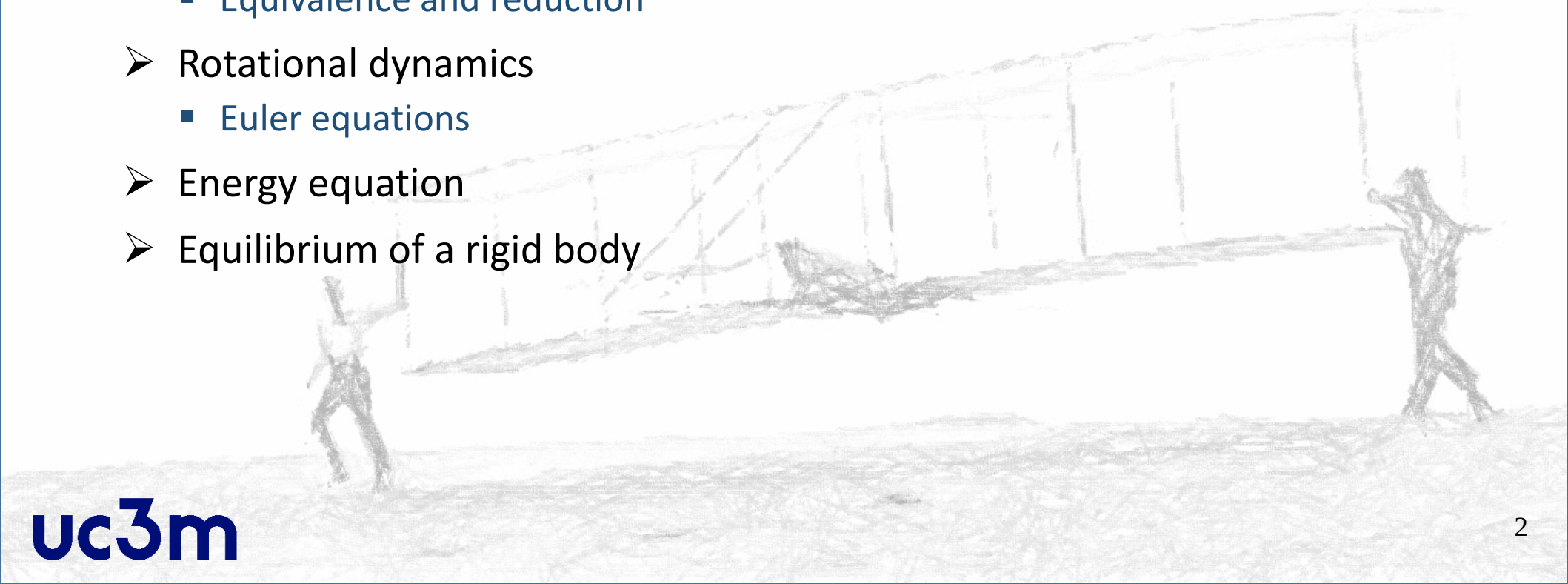
Course 251-14165 – BSc Aerospace Engineering

10. Rigid body dynamics



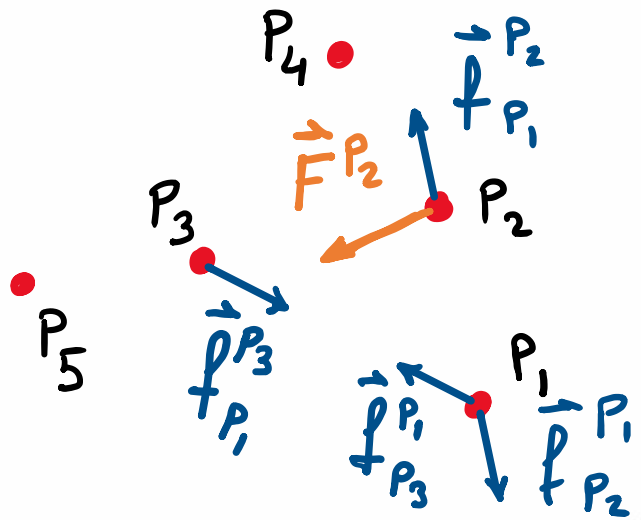
Contents

- Introduction
- A system of particles
  - Linear momentum equation
  - Angular momentum equation
  - Energy equation
- Application to rigid bodies
- Systems of forces and torques
  - Equivalence and reduction
- Rotational dynamics
  - Euler equations
- Energy equation
- Equilibrium of a rigid body



Introduction

- We will first consider a system composed of N free particles
  - The system has $3N$ degrees of freedom
  - We have $3N$ scalar equations, from $m^{P_i}\boldsymbol{a}_0^{P_i} = \boldsymbol{F}^{P_i} + \sum_{j\neq i}\boldsymbol{f}_{P_j}^{P_i}$ for $i = 1, \ldots, N$



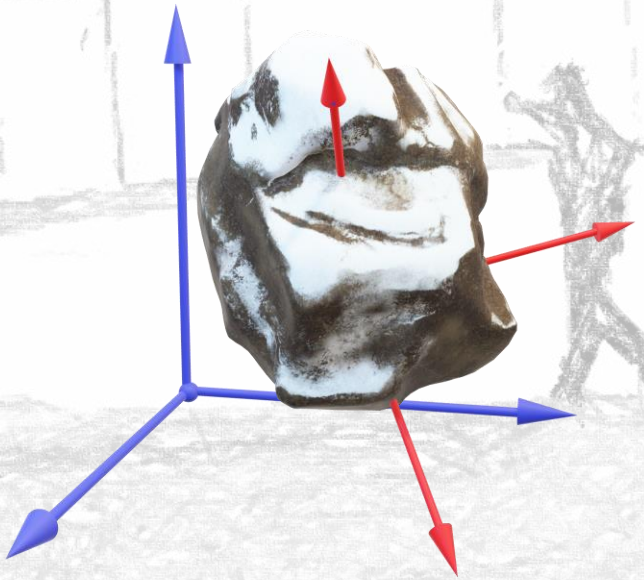


Internal forces
($\boldsymbol{f}_{P_j}^{P_i} = -\boldsymbol{f}_{P_i}^{P_j}$)



External forces

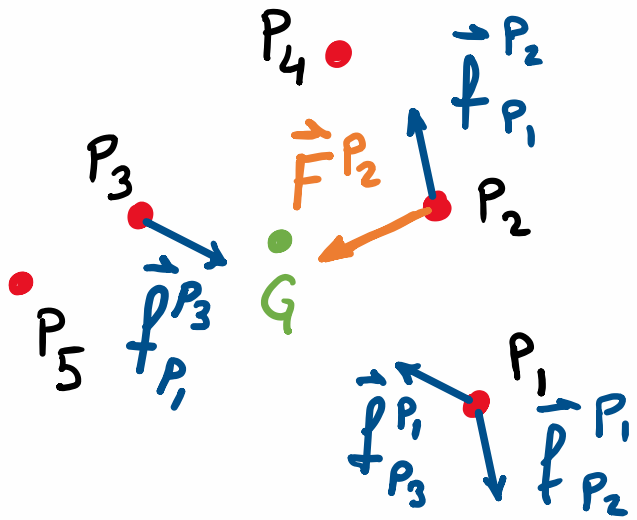
- A rigid body can be thought of as an infinite collection of particles of differential mass. However, the rigid body constraint is a very strong one: there are only 6 degrees of freedom, so we just need 6 linearly-independent scalar equations!



Linear momentum equation

- For N unconstrained particles, we need the $3N$ scalar equations:

$$m^{P_1}\boldsymbol{a}_0^{P_1} = \boldsymbol{F}^{P_1} + \boldsymbol{f}_{P_2}^{P_1} + \boldsymbol{f}_{P_3}^{P_1} + \cdots$$
$$m^{P_2}\boldsymbol{a}_0^{P_2} = \boldsymbol{F}^{P_2} + \boldsymbol{f}_{P_1}^{P_2} + \boldsymbol{f}_{P_3}^{P_2} + \cdots$$
$$m^{P_3}\boldsymbol{a}_0^{P_2} = \boldsymbol{F}^{P_3} + \boldsymbol{f}_{P_1}^{P_3} + \boldsymbol{f}_{P_2}^{P_3} + \cdots$$
$$\ldots = \cdots$$



That said, we can construct some useful global equations:

- Summing over, all internal forces cancel out
- Introducing the *resultant external force* as

$$\boldsymbol{F} = \sum \boldsymbol{F}^{P_i}$$

- The equation for the total linear momentum of the system (dynamics of the center of mass) reads:

$$\left.\frac{\mathrm{d}\boldsymbol{p}_0}{\mathrm{d}t}\right|_0 = m\boldsymbol{a}_0^G = \boldsymbol{F}$$

$$m\boldsymbol{r}_0^G = \sum_i m^{P_i}\boldsymbol{r}_0^{P_i}$$
$$m\boldsymbol{v}_0^G = \sum_i m^{P_i}\boldsymbol{v}_0^{P_i}$$
$$m\boldsymbol{a}_0^G = \sum_i m^{P_i}\boldsymbol{a}_0^{P_i}$$

The dynamics of the center of mass of the system are equivalent to those of a single point particle with the total mass of the system, subjected to the resultant force

Angular momentum equation

- Taking the cross product with $\boldsymbol{AP}_i$:

$$m^{P_1}\boldsymbol{AP}_1 \times \boldsymbol{a}_0^{P_1} = \boldsymbol{AP}_1 \times \boldsymbol{F}^{P_1} + \boldsymbol{AP}_1 \times \boldsymbol{f}_{P_2}^{P_1} + \boldsymbol{AP}_1 \times \boldsymbol{f}_{P_3}^{P_1} + \cdots$$
$$m^{P_2}\boldsymbol{AP}_2 \times \boldsymbol{a}_0^{P_2} = \boldsymbol{AP}_2 \times \boldsymbol{F}^{P_2} + \boldsymbol{AP}_2 \times \boldsymbol{f}_{P_1}^{P_2} + \boldsymbol{AP}_2 \times \boldsymbol{f}_{P_3}^{P_2} + \cdots$$
$$m^{P_3}\boldsymbol{AP}_3 \times \boldsymbol{a}_0^{P_2} = \boldsymbol{AP}_3 \times \boldsymbol{F}^{P_3} + \boldsymbol{AP}_3 \times \boldsymbol{f}_{P_1}^{P_3} + \boldsymbol{AP}_3 \times \boldsymbol{f}_{P_2}^{P_3} + \cdots$$
$$\ldots = \cdots$$

- Summing over, all internal torques cancel out
- We introduce the *resultant external torque about A* as

$$\boldsymbol{M}_A = \sum \boldsymbol{AP}_i \times \boldsymbol{F}^{P_i}$$

- Note that

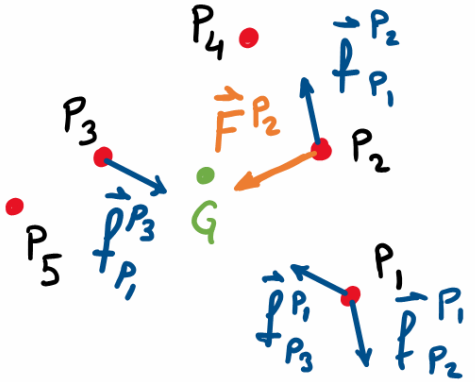
$$m^{P_i}\boldsymbol{AP}_i \times \boldsymbol{a}_0^{P_i} = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_0 \left(m^{P_i}\boldsymbol{AP}_i \times \boldsymbol{v}_0^{P_i}\right) - m^{P_i}\left(\boldsymbol{v}_0^{P_i} - \boldsymbol{v}_0^{A}\right) \times \boldsymbol{v}_0^{P_i}$$

$$\sum m^{P_i}\boldsymbol{AP}_i \times \boldsymbol{a}_0^{P_i} = \frac{\mathrm{d}\boldsymbol{H}_{A0}}{\mathrm{d}t}\bigg|_0 + m\boldsymbol{v}_0^{A} \times \boldsymbol{v}_0^{G} = \boldsymbol{M}_A$$

Kinetic energy equation

- Taking the dot product with $\boldsymbol{v}_0^{P_i}$:

$$m^{P_1}\boldsymbol{v}_0^{P_1}\cdot\boldsymbol{a}_0^{P_1}=\boldsymbol{v}_0^{P_1}\cdot\boldsymbol{F}^{P_1}+\boldsymbol{v}_0^{P_1}\cdot\boldsymbol{f}_{P_2}^{P_1}+\boldsymbol{v}_0^{P_1}\cdot\boldsymbol{f}_{P_3}^{P_1}+\cdots$$
$$m^{P_2}\boldsymbol{v}_0^{P_2}\cdot\boldsymbol{a}_0^{P_2}=\boldsymbol{v}_0^{P_2}\cdot\boldsymbol{F}^{P_2}+\boldsymbol{v}_0^{P_2}\cdot\boldsymbol{f}_{P_1}^{P_2}+\boldsymbol{v}_0^{P_2}\cdot\boldsymbol{f}_{P_3}^{P_2}+\cdots$$
$$m^{P_3}\boldsymbol{v}_0^{P_3}\cdot\boldsymbol{a}_0^{P_2}=\boldsymbol{v}_0^{P_3}\cdot\boldsymbol{F}^{P_3}+\boldsymbol{v}_0^{P_3}\cdot\boldsymbol{f}_{P_1}^{P_3}+\boldsymbol{v}_0^{P_3}\cdot\boldsymbol{f}_{P_2}^{P_3}+\cdots$$
$$\ldots=\cdots$$



- When summing over, the power associated to internal forces <u>does not cancel out in general</u>!
  - However, for a rigid body, since the distance between P_i and P_j remains constant, then:

$$\boldsymbol{v}_0^{P_i}\cdot\boldsymbol{f}_{P_j}^{P_i}=-\boldsymbol{v}_0^{P_j}\cdot\boldsymbol{f}_{P_i}^{P_j}$$

- Note that

$$m^{P_i}\boldsymbol{v}_0^{P_i}\cdot\boldsymbol{a}_0^{P_i}=m^{P_i}\boldsymbol{v}_0^{P_i}\cdot\left.\frac{\mathrm{d}\boldsymbol{v}_0^{P_i}}{\mathrm{d}t}\right|_0=\frac{1}{2}m^{P_i}\left.\frac{\mathrm{d}\left(v_0^{P_i}\right)^2}{\mathrm{d}t}\right|_0=\left.\frac{\mathrm{d}T_0^{P_i}}{\mathrm{d}t}\right|_0$$

- Defining the *resultant power* of external and internal forces as $\dot{W}$ and $\dot{w}$,

$$\left.\frac{\mathrm{d}T_0}{\mathrm{d}t}\right|_0=\dot{W}_0+\dot{w}_0$$

External *and* internal forces may exert work, and may be conservative or non-conservative

Interlude: on potential energy of internal forces

- Internal forces, just like external forces, might be conservative. This requires that a function

$$V = V\left(\boldsymbol{r}_0^{P_i}, \boldsymbol{r}_0^{P_j}\right)$$

exists, so that we can write the pair $\boldsymbol{f}_{P_j}^{P_i}, \boldsymbol{f}_{P_i}^{P_j}$ as (e.g. in Cartesian coordinates):

$$\boldsymbol{f}_{P_j}^{P_i} = -\frac{\partial V}{\partial x^{P_i}}\boldsymbol{i} - \frac{\partial V}{\partial y^{P_i}}\boldsymbol{j} - \frac{\partial V}{\partial z^{P_i}}\boldsymbol{k}$$

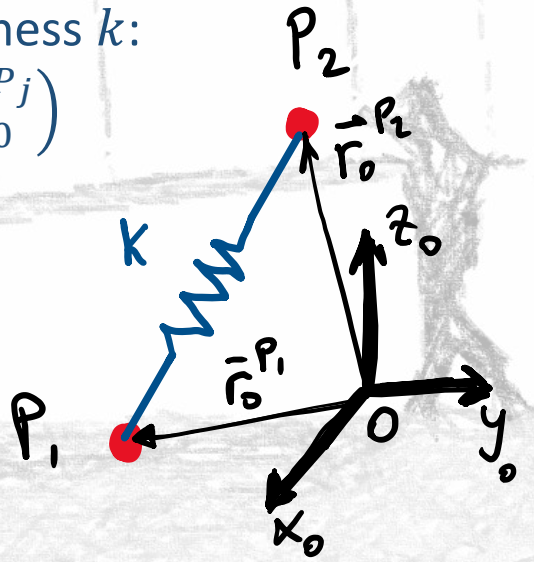
$$\boldsymbol{f}_{P_i}^{P_j} = -\frac{\partial V}{\partial x^{P_j}}\boldsymbol{i} - \frac{\partial V}{\partial y^{P_j}}\boldsymbol{j} - \frac{\partial V}{\partial z^{P_j}}\boldsymbol{k}$$

  - Example: if P_1 and P_2 are connected by a spring of stiffness k:

$$\boldsymbol{f}_{P_2}^{P_1} = -k\left(\boldsymbol{r}_0^{P_i} - \boldsymbol{r}_0^{P_j}\right); \qquad \boldsymbol{f}_{P_1}^{P_2} = +k\left(\boldsymbol{r}_0^{P_i} - \boldsymbol{r}_0^{P_j}\right)$$

    The potential energy function for this system is:

$$V = \frac{1}{2}k\left|\boldsymbol{r}_0^{P_i} - \boldsymbol{r}_0^{P_j}\right|^2$$



Caution! when writing the energy equation, each internal potential energy term appears only once in total (not once per particle)

Dynamics of rigid bodies

- The rigid body constraint makes most of the constituent particle equations redundant. We only need 6 independent scalar equations!
  - The equation of translational motion for the center of mass:

$$m\boldsymbol{a}_0^G = \boldsymbol{F}$$

The motion of the center of mass depends only on the resultant external force and the total mass of the body

  - The equation of rotational motion about a point A:

$$\left.\frac{\mathrm{d}\boldsymbol{H}_{A0}}{\mathrm{d}t}\right|_0 = \boldsymbol{M}_A - m\boldsymbol{v}_0^A \times \boldsymbol{v}_0^G \qquad \boldsymbol{H}_{A0} = \bar{\bar{\boldsymbol{I}}}_A \cdot \boldsymbol{\omega}_{b0} + m\boldsymbol{AG} \times \boldsymbol{v}_0^{A(b)}$$

    - Naturally, we will always choose A to be a stationary body point ($\boldsymbol{v}_0^A = \boldsymbol{v}_0^{A(b)} = \boldsymbol{0}$). Or, we will pick $A \equiv G$. In either case:

$$\left.\frac{\mathrm{d}\left(\bar{\bar{\boldsymbol{I}}}_A \cdot \boldsymbol{\omega}_{b0}\right)}{\mathrm{d}t}\right|_0 = \boldsymbol{M}_A$$

The rotational motion depends only on the resultant external torque and the tensor of inertia of the rigid body

- The equation of energy is not an independent equation. And, the internal forces do not exert work in a rigid body

$$\left.\frac{\mathrm{d}T_0}{\mathrm{d}t}\right|_0 = \dot{W}_0$$

Systems of forces and torques

These ideas are only applicable in the context of rigid bodies

- Consider a rigid body subjected to forces $\boldsymbol{F}_P$, $\boldsymbol{F}_Q$, ... on points P, Q, ...
- We say the body is under a *system of external forces*

$$\{\boldsymbol{F}_P, \boldsymbol{F}_Q, \dots\}$$

- The dynamics of the rigid body are seen to depend only on the resultant force and the resultant torque about a point A

$$\boldsymbol{F} = \boldsymbol{F}_P + \boldsymbol{F}_Q + \cdots$$
$$\boldsymbol{M}_A = \boldsymbol{AP} \times \boldsymbol{F}_P + \boldsymbol{AQ} \times \boldsymbol{F}_Q + \cdots$$

- We say the system of forces $\{\boldsymbol{F}_P, \boldsymbol{F}_Q, \dots\}$ can be reduced to the force-torque system about A:

$$\{\boldsymbol{F}; \boldsymbol{M}_A\}$$

  - Observe that the force $\boldsymbol{F}$ is independent of the point A that we choose
  - However, $\boldsymbol{M}_A$ depends on A

Two systems of forces are equivalent if they reduce to the same force-torque system about a given point

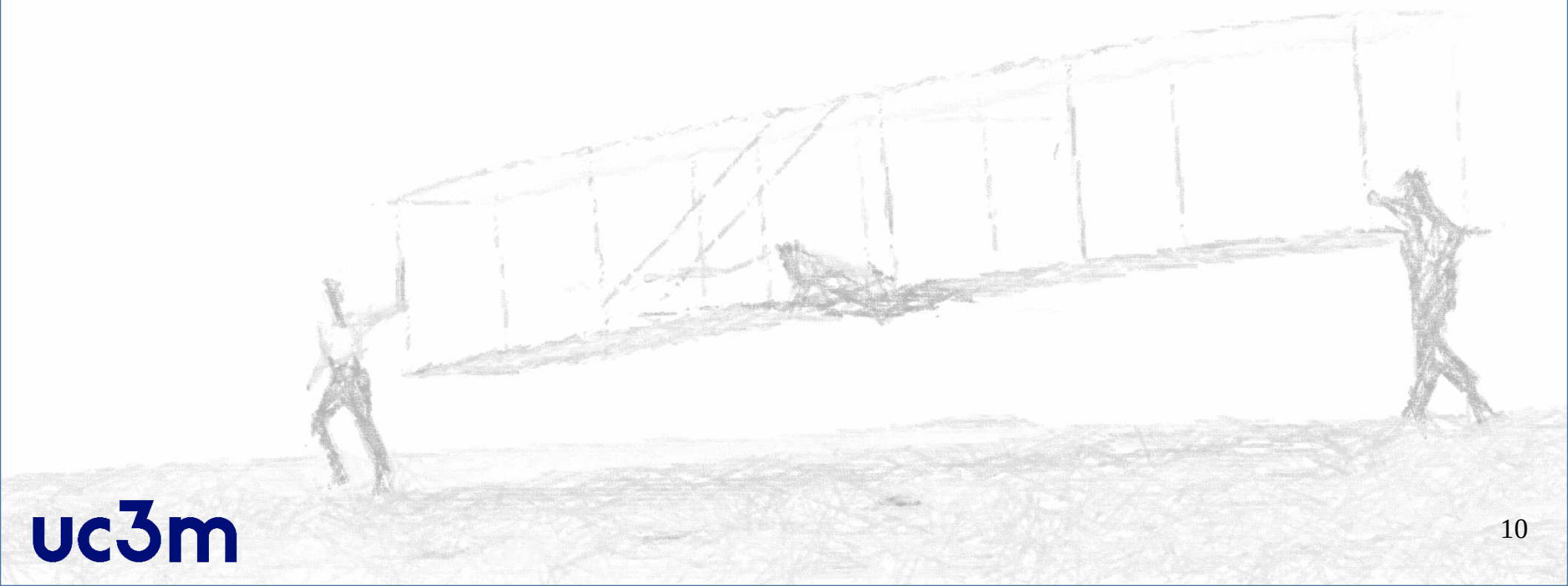
Systems of forces and torques

- Given a reduction of a force and torque system $\{\boldsymbol{F}; \boldsymbol{M}_A\}$ (torque computed at a point A), can we "translate" it to a point B?

$$\boldsymbol{F} = \boldsymbol{F}$$

$$\boldsymbol{M}_B = \boldsymbol{M}_A + \boldsymbol{BA} \times \boldsymbol{F} = \boldsymbol{M}_A + \boldsymbol{F} \times \boldsymbol{AB}$$

- The new reduction is $\{\boldsymbol{F}; \boldsymbol{M}_B\}$



Systems of forces and torques

- The power done by $\{\boldsymbol{F}_P, \boldsymbol{F}_Q, \ldots\}$ is

$$\dot{W}_0 = \boldsymbol{F}_P \cdot \boldsymbol{v}_0^P + \boldsymbol{F}_Q \cdot \boldsymbol{v}_0^Q + \cdots$$

- But $\boldsymbol{v}_0^P = \boldsymbol{v}_0^A + \boldsymbol{\omega}_{b0} \times \boldsymbol{AP}$, so:

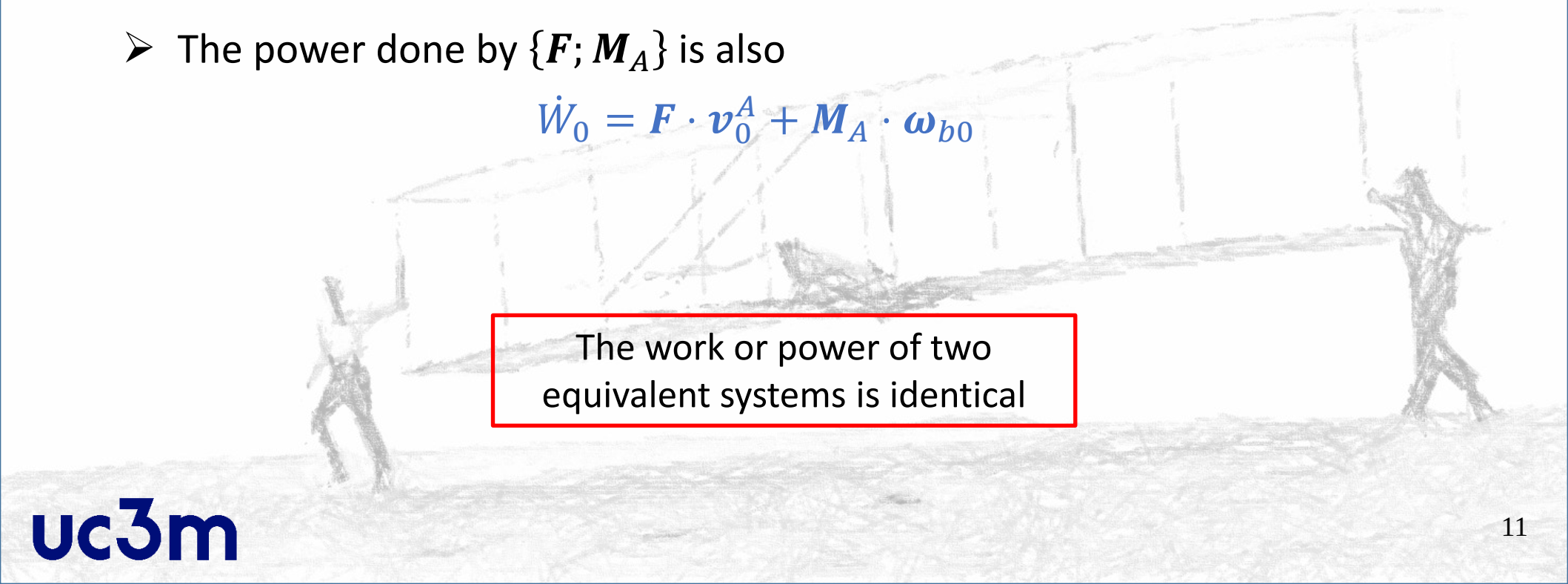
$$\dot{W}_0 = \boldsymbol{F}_P \cdot \boldsymbol{v}_0^A + \boldsymbol{F}_Q \cdot \boldsymbol{v}_0^A + \cdots + \boldsymbol{F}_P \cdot (\boldsymbol{\omega}_{b0} \times \boldsymbol{AP}) + \boldsymbol{F}_Q \cdot (\boldsymbol{\omega}_{b0} \times \boldsymbol{AQ}) + \cdots$$

- However, $\boldsymbol{F}_P \cdot (\boldsymbol{\omega}_{b0} \times \boldsymbol{AP}) = \boldsymbol{\omega}_{b0} \cdot (\boldsymbol{AP} \times \boldsymbol{F}_P)$. Therefore:

$$\dot{W}_0 = \boldsymbol{F} \cdot \boldsymbol{v}_0^A + \boldsymbol{M}_A \cdot \boldsymbol{\omega}_{b0}$$

- The power done by $\{\boldsymbol{F}; \boldsymbol{M}_A\}$ is also

$$\dot{W}_0 = \boldsymbol{F} \cdot \boldsymbol{v}_0^A + \boldsymbol{M}_A \cdot \boldsymbol{\omega}_{b0}$$



The work or power of two equivalent systems is identical

Systems of forces and torques

- A more general approach considers that rigid bodies can be subjected to a system of external forces and torques (these can be thought of as *pairs of forces*)

$$\{\boldsymbol{F}_P, \boldsymbol{F}_Q, \dots; \boldsymbol{Q}_1, \boldsymbol{Q}_2, \dots\}$$

- In this case, the system of forces and torques is reduced to $\{\boldsymbol{F}; \boldsymbol{M}_A\}$ with:

$$\boldsymbol{F} = \boldsymbol{F}_P + \boldsymbol{F}_Q + \cdots$$
$$\boldsymbol{M}_A = \boldsymbol{AP} \times \boldsymbol{F}_P + \boldsymbol{AQ} \times \boldsymbol{F}_Q + \cdots + \boldsymbol{Q}_1 + \boldsymbol{Q}_2 + \cdots$$

- The power associated to this system is

$$\dot{W}_0 = \boldsymbol{F}_P \cdot \boldsymbol{v}_0^P + \boldsymbol{F}_Q \cdot \boldsymbol{v}_0^Q + \cdots + \boldsymbol{Q}_1 \cdot \boldsymbol{\omega}_{b0} + \boldsymbol{Q}_2 \cdot \boldsymbol{\omega}_{b0} + \cdots$$

and it still reduces to

$$\dot{W}_0 = \boldsymbol{F} \cdot \boldsymbol{v}_0^A + \boldsymbol{M}_A \cdot \boldsymbol{\omega}_{b0}$$

Rotational dynamics

- The translational dynamics equation is identical to the equation of a point particle, so there is nothing new there. However, the rotational equation is new and deserves our attention.
- Provided that A is a stationary body point or the center of mass,

$$\left.\frac{\mathrm{d}\left(\bar{\bar{I}}_A \cdot \boldsymbol{\omega}_{b0}\right)}{\mathrm{d}t}\right|_0 = \boldsymbol{M}_A$$

- The tensor of inertia may not be constant in S_0, but it is in a body-fixed reference frame S_b. Invoking Coriolis' theorem:

$$\left.\frac{\mathrm{d}\left(\bar{\bar{I}}_A \cdot \boldsymbol{\omega}_{b0}\right)}{\mathrm{d}t}\right|_0 = \bar{\bar{I}}_A \cdot \left.\frac{\mathrm{d}\boldsymbol{\omega}_{b0}}{\mathrm{d}t}\right|_b + \boldsymbol{\omega}_{b0} \times \left(\bar{\bar{I}}_A \cdot \boldsymbol{\omega}_{b0}\right) = \boldsymbol{M}_A$$

$$\boxed{\bar{\bar{I}}_A \cdot \boldsymbol{\alpha}_{b0} = \boldsymbol{M}_A - \boldsymbol{\omega}_{b0} \times \left(\bar{\bar{I}}_A \cdot \boldsymbol{\omega}_{b0}\right)}$$

- This way of writing the rotational dynamics equation (i.e. in the body-fixed reference frame) is known as *Euler's equation*

Rotational dynamics

$$\bar{\bar{I}}_A \cdot \boldsymbol{\alpha}_{b0} = \boldsymbol{M}_A - \boldsymbol{\omega}_{b0} \times \left(\bar{\bar{I}}_A \cdot \boldsymbol{\omega}_{b0}\right)$$

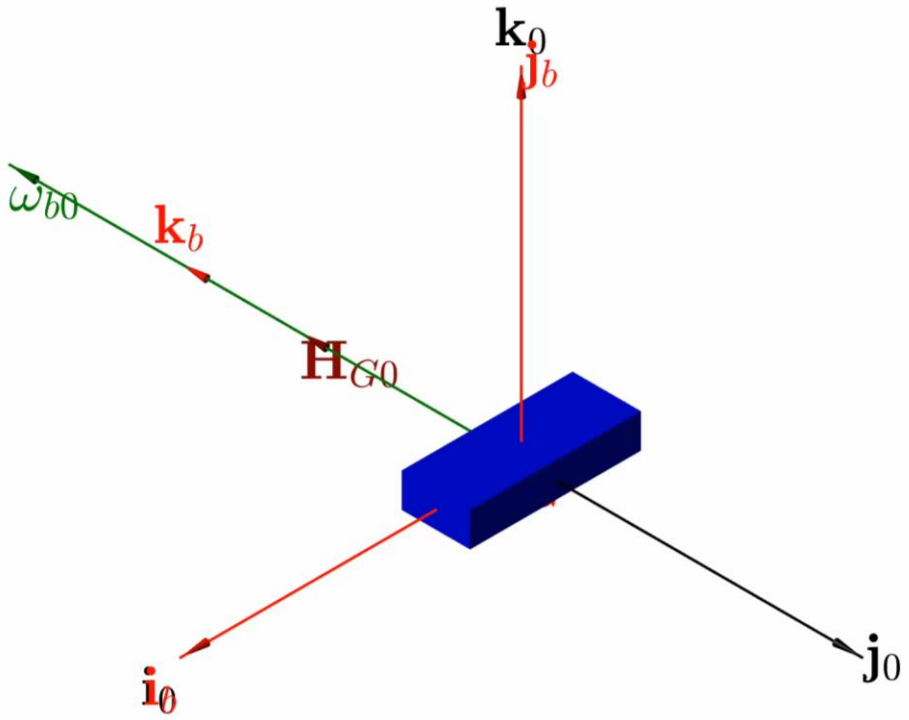
- Let us project this vector equation in the directions of a (body-fixed) principal basis of inertia, $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ where the principal moments of inertia are I_1, I_2, I_3:

$$I_1\alpha_1 = M_1 + (I_2 - I_3)\omega_2\omega_3$$
$$I_2\alpha_2 = M_2 + (I_3 - I_1)\omega_1\omega_3$$
$$I_3\alpha_3 = M_3 + (I_1 - I_2)\omega_1\omega_2$$

- Then:
  - If the body is rotating about a principal axis of inertia, say 3, then $\omega_1 = \omega_2 = 0$ and the equations are decoupled. This is a "clean" rotation about a principal axis
  - If the body has, say, $\omega_1, \omega_2 \neq 0$ and the principal moments of inertia are all different, then the rotations about the three principal axes are all coupled. This is a "dirty" rotation.

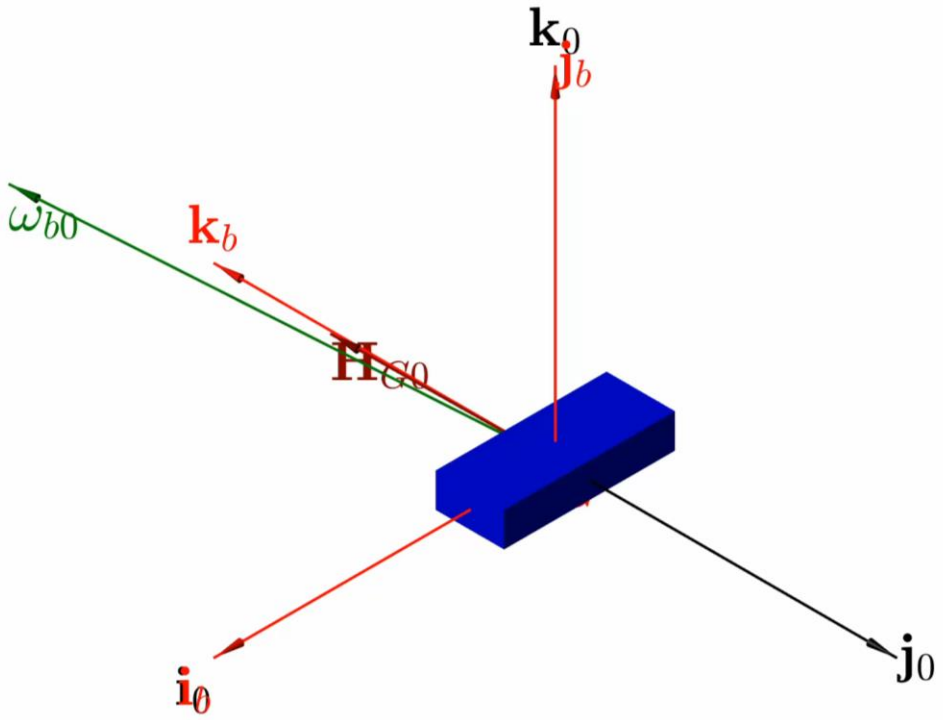
Note that $\omega_1, \omega_2, \omega_3$ are the components of $\boldsymbol{\omega}_{b0}$ in the (body-fixed) reference frame with the principal directions of inertia

Rotational dynamics



"Clean" rotation

Rotational dynamics



"Dirty" rotation

Energy equation

- Consider a body b subjected to a system of forces:

$$\{\boldsymbol{F}_P, \boldsymbol{F}_Q, \dots; \boldsymbol{Q}_1, \boldsymbol{Q}_2, \dots\} \sim \{\boldsymbol{F}; \boldsymbol{M}_A\}$$

- We can evaluate the power or work done on the body by looking at the individual forces/torques:

$$\left.\frac{\mathrm{d}T_0}{\mathrm{d}t}\right|_0 = \boldsymbol{F}_P \cdot \boldsymbol{v}_0^P + \cdots + \boldsymbol{Q}_1 \cdot \boldsymbol{\omega}_{b0} + \cdots$$

  - External forces $\boldsymbol{F}_P$ that are perpendicular to the velocity of their point P of application (any point on their line of application, actually) do zero work
  - External torques $\boldsymbol{Q}_1$ that are perpendicular to the angular velocity vector $\boldsymbol{\omega}_{b0}$ do zero work
  - A force or torque is *conservative* if the work done by it (as the body moves from a reference configuration to a new configuration) can be expressed as a function $-V$ of the generalized coordinates only.
    - Then, the force or torque can be expressed as a function of the partial derivatives of $-V$ with respect to the generalized coordinates

Energy equation

- Alternatively, we may compute the power or work with the resultant $\boldsymbol{F}$ and $\boldsymbol{M}_A$ about a point A:

$$\left.\frac{\mathrm{d}T_0}{\mathrm{d}t}\right|_0 = \boldsymbol{F}\cdot\boldsymbol{v}_0^A + \boldsymbol{M}_A\cdot\boldsymbol{\omega}_{b0}$$

  - Same comments apply with respect to the work of $\boldsymbol{F}$ and $\boldsymbol{M}_A$

- Conservation of mechanical energy for rigid bodies:

When all forces/torques that exert work are conservative, the *mechanical energy of the rigid body* is conserved:

$$T_0 + V_0 = E_0$$

Some of the common potential energy functions (e.g. gravity force) must be computed as integrals over the rigid body.

Equilibrium of a rigid body

- The generalization of "equilibrium configuration" and "stability" to rigid bodies is straightforward:
  - We say a configuration (position + attitude) is an *equilibrium* if, when we leave the body in this configuration with zero translational and rotational velocity, it remains at rest indefinitely. This means that the rigid body acceleration and angular acceleration must be zero:

$$\boldsymbol{a}_0^{\mathrm{P}} = \mathbf{0}; \qquad \boldsymbol{\alpha}_{b0} = \mathbf{0}$$

  - From the equations of motion, this occurs iff the resultant force and torque about one point (say, A) are zero:

$$\boldsymbol{F} = \mathbf{0}; \qquad \boldsymbol{M}_A = \mathbf{0}$$

  - Equilibrium configurations are *stable* if, when we perturb the position or attitude or the velocity/rotation of the body slightly with respect to that configuration, the body does not depart far from it.
  - Taking for example a body point C and 3-1-3 Euler angles, we say that a configuration defined by $\boldsymbol{r}_0^C = \boldsymbol{r}_0^*$ and $\psi_0, \theta_0, \varphi_0$ is stable iff $\forall \varepsilon_r, \varepsilon_v, \varepsilon_a, \varepsilon_\omega > 0$, $\exists \delta_r, \delta_v, \delta_a, \delta_\omega > 0$ such that if we perturb b with: $|\boldsymbol{r}_0^C - \boldsymbol{r}_0^*| < \delta_r$, $|\boldsymbol{v}_0^C| < \delta_v$, $|\psi - \psi_0|, |\theta - \theta_0|, |\varphi - \varphi_0| < \delta_a$ and $|\boldsymbol{\omega}_{b0}| < \delta_\omega$ then $|\boldsymbol{r}_0^C - \boldsymbol{r}_0^*| < \varepsilon_r$, $|\boldsymbol{v}_0^C| < \varepsilon_v$, $|\psi - \psi_0|, |\theta - \theta_0|, |\varphi - \varphi_0| < \varepsilon_a$ and $|\boldsymbol{\omega}_{b0}| < \varepsilon_\omega$ for all $t > 0$.

Thank you!

