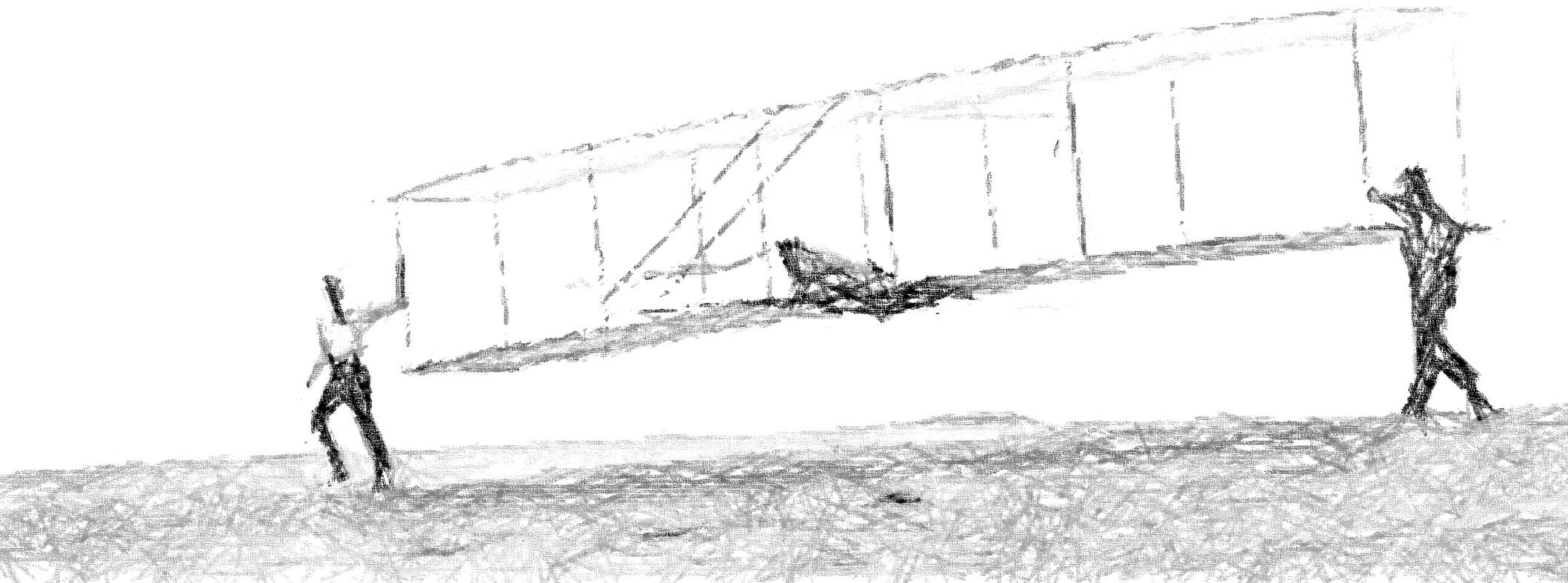
uc3m

Mechanics applied to Aerospace Engineering

Course 251-14165 – BSc Aerospace Engineering

9. Momentum and energy



Contents

- Linear momentum
- Angular momentum
- Kinetic energy
- Summary of useful relations
- A first outlook into the dynamics of rigid bodies

Just an advance of what we will see next week



Linear momentum

- We define the *linear momentum of a rigid body* as the sum (integral) of the linear momenta of all its constituent particles:

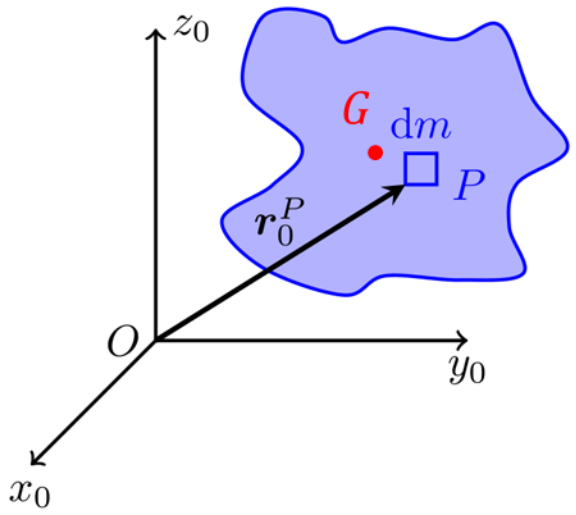
$$\boldsymbol{p}_0 = \int_{\mathcal{D}} \mathrm{d}\boldsymbol{p}_0 = \int_{\mathcal{D}} \boldsymbol{v}_0^P \, \mathrm{d}m$$

- Recalling the definition of the center of mass (from the first moment of the mass distribution) and taking derivatives:

$$m\boldsymbol{r}_0^G = \int_{\mathcal{D}} \boldsymbol{r}_0^P \, \mathrm{d}m.$$

$$m\boldsymbol{v}_0^G = \int_{\mathcal{D}} \boldsymbol{v}_0^P \, \mathrm{d}m = \boldsymbol{p}_0$$

  - The linear momentum of the body is equal to the linear momentum of a point particle with the same total mass, located and moving like the center of mass



Angular momentum

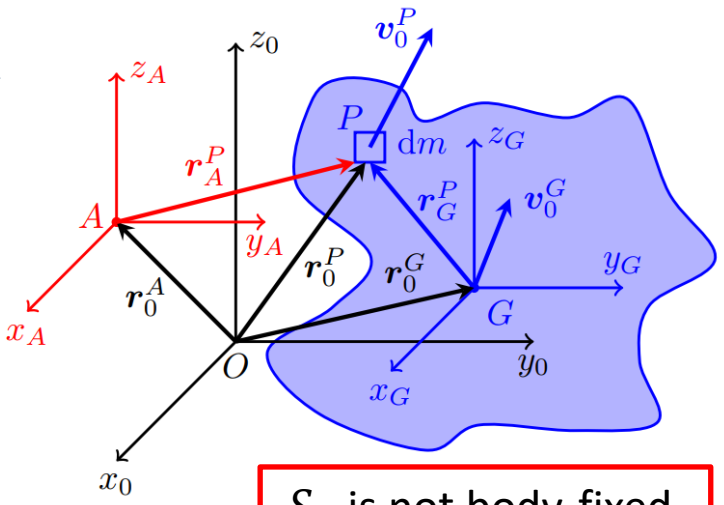
- We define the *angular momentum of a rigid body* about a point A as the sum (integral) of the angular momenta of all its constituent particles about that point:

$$\boldsymbol{H}_{A,0} = \int_{\mathcal{D}} \mathrm{d}\boldsymbol{H}_{A,0} = \int_{\mathcal{D}} \boldsymbol{r}_A^P \times \boldsymbol{v}_0^P \mathrm{d}m \equiv \int_D \boldsymbol{AP} \times \boldsymbol{v}_0^P \mathrm{d}m$$

- To expand this integral we will use the velocity field equation of the rigid body:

$$\boldsymbol{v}_0^P = \boldsymbol{v}_0^{A(B)} + \boldsymbol{\omega}_{B0} \times \boldsymbol{r}_A^P$$

where $A(B)$ is the point of the body that happens to be on top of point A, at this instant



S_G is not body-fixed, in general

$$\boldsymbol{H}_{A,0} = \left(\int_{\mathcal{D}} \boldsymbol{r}_A^P \mathrm{d}m\right) \times \boldsymbol{v}_0^{A(B)} + \int_{\mathcal{D}} \boldsymbol{r}_A^P \times (\boldsymbol{\omega}_{B0} \times \boldsymbol{r}_A^P) \mathrm{d}m$$

$$m\boldsymbol{r}_A^G \times \boldsymbol{v}_0^{A(B)} \qquad \int_{\mathcal{D}} \left[(r_A^P)^2 \boldsymbol{\omega}_{B0} - (\boldsymbol{r}_A^P \cdot \boldsymbol{\omega}_{B0}) \boldsymbol{r}_A^P\right] \mathrm{d}m$$

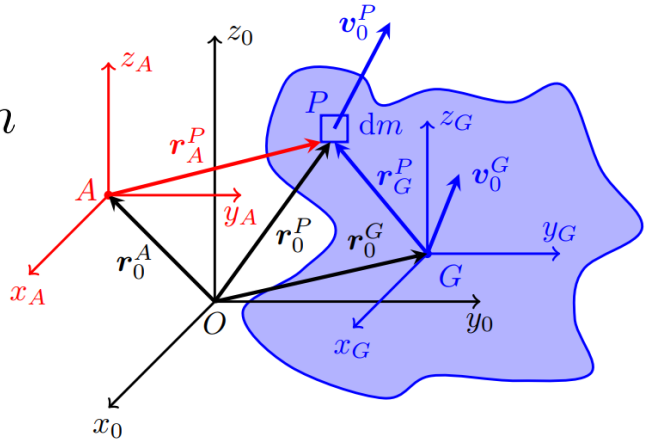
$$(r_A^P)^2 \boldsymbol{\omega}_{B0} = (r_A^P)^2 \overline{\overline{\mathcal{U}}} \cdot \boldsymbol{\omega}_{B0} \qquad \boldsymbol{r}_A^P (\boldsymbol{r}_A^P \cdot \boldsymbol{\omega}_{B0}) = (\boldsymbol{r}_A^P \otimes \boldsymbol{r}_A^P) \cdot \boldsymbol{\omega}_{B0}$$

Angular momentum

$$\boldsymbol{H}_{A,0} = \left(\int_{\mathcal{D}} \boldsymbol{r}_A^P \mathrm{d}m\right) \times \boldsymbol{v}_0^{A(B)} + \int_{\mathcal{D}} \boldsymbol{r}_A^P \times (\boldsymbol{\omega}_{B0} \times \boldsymbol{r}_A^P)\mathrm{d}m$$

$$m\boldsymbol{r}_A^G \times \boldsymbol{v}_0^{A(B)}$$

$$\int_{\mathcal{D}} \left[(r_A^P)^2 \overline{\overline{\mathcal{U}}} \cdot \boldsymbol{\omega}_{B0} - (\boldsymbol{r}_A^P \otimes \boldsymbol{r}_A^P) \cdot \boldsymbol{\omega}_{B0}\right] \mathrm{d}m = \overline{\overline{\mathcal{I}}}_A \cdot \boldsymbol{\omega}_{B0}$$



- The angular momentum of a rigid body can be express as:

$$\boldsymbol{H}_{A,0} = \overline{\overline{\mathcal{I}}}_A \cdot \boldsymbol{\omega}_{B0} + m\boldsymbol{r}_A^G \times \boldsymbol{v}_0^{A(B)}$$

- Observe that this equation can be simplified if...
  - $A(B)$ is a body point that does not move, i.e., it is fixed in S_0:

$$\boldsymbol{H}_{A,0} = \overline{\overline{\mathcal{I}}}_A \cdot \boldsymbol{\omega}_{B0}$$

  - We choose the center of mass G as the reference point about which we compute the angular momentum:

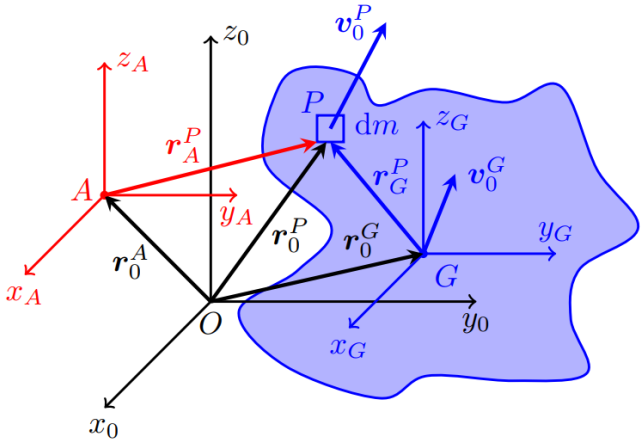
$$\boldsymbol{H}_{G,0} = \overline{\overline{\mathcal{I}}}_G \cdot \boldsymbol{\omega}_{B0}$$

Angular momentum

- A different expansion leads to a useful relation:

$$\boldsymbol{H}_{A,0} = \int_{\mathcal{D}} \mathrm{d}\boldsymbol{H}_{A,0} = \int_{\mathcal{D}} \boldsymbol{r}_A^P \times \boldsymbol{v}_0^P \mathrm{d}m$$

$$\boldsymbol{r}_A^P = \boldsymbol{r}_A^G + \boldsymbol{r}_G^P \qquad \boldsymbol{v}_0^P = \boldsymbol{v}_0^G + \boldsymbol{\omega}_{B0} \times \boldsymbol{r}_G^P$$



$$\boldsymbol{H}_{A,0} = \int_{\mathcal{D}} (\boldsymbol{r}_A^G + \boldsymbol{r}_G^P) \times (\boldsymbol{v}_0^G + \boldsymbol{\omega}_{B0} \times \boldsymbol{r}_G^P) \mathrm{d}m$$

$$= \int_{\mathcal{D}} \boldsymbol{r}_A^G \times \boldsymbol{v}_0^G \mathrm{d}m \qquad m\boldsymbol{r}_A^G \times \boldsymbol{v}_0^G$$

$$+ \int_{\mathcal{D}} \boldsymbol{r}_G^P \times \boldsymbol{v}_0^G \mathrm{d}m \qquad \left(\int_{\mathcal{D}} \boldsymbol{r}_G^P \mathrm{d}m \right) \times \boldsymbol{v}_0^G = \boldsymbol{0}$$

$$+ \int_{\mathcal{D}} \boldsymbol{r}_A^G \times (\boldsymbol{\omega}_{B0} \times \boldsymbol{r}_G^P) \mathrm{d}m \qquad \boldsymbol{r}_A^G \times \left(\boldsymbol{\omega}_{B0} \times \int_{\mathcal{D}} \boldsymbol{r}_G^P \mathrm{d}m \right) = \boldsymbol{0}$$

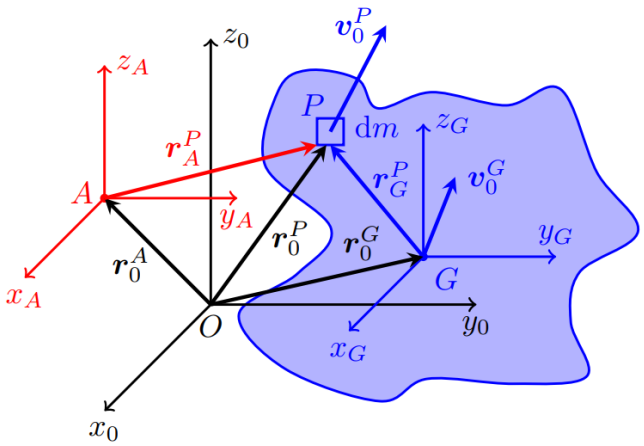
$$+ \int_{\mathcal{D}} \boldsymbol{r}_G^P \times (\boldsymbol{\omega}_{B0} \times \boldsymbol{r}_G^P) \mathrm{d}m \qquad \int_{\mathcal{D}} \boldsymbol{r}_G^P \times \boldsymbol{v}_G^P \mathrm{d}m = \boldsymbol{H}_{G,G}$$

$$\boldsymbol{v}_G^P = \boldsymbol{\omega}_{B0} \times \boldsymbol{r}_G^P$$

Angular momentum

- However:

$$\boldsymbol{H}_{G,G} = \int_{\mathcal{D}} \boldsymbol{r}_G^P \times \boldsymbol{v}_G^P \mathrm{d}m = \int_{\mathcal{D}} \boldsymbol{r}_G^P \times (\boldsymbol{v}_0^P - \boldsymbol{v}_0^G)\mathrm{d}m$$
$$= \int_{\mathcal{D}} \boldsymbol{r}_G^P \times \boldsymbol{v}_0^P \mathrm{d}m - \left(\int_{\mathcal{D}} \boldsymbol{r}_G^P \mathrm{d}m\right) \times \boldsymbol{v}_0^G = \boldsymbol{H}_{G,0} + \boldsymbol{0}$$



So:

$$\boldsymbol{H}_{G,G} \equiv \boldsymbol{H}_{G,0}$$

- Finally, we find the relation between the angular momentum about a point A and the center of mass G:

$$\boldsymbol{H}_{A,0} = \boldsymbol{H}_{G,0} + m\boldsymbol{r}_A^G \times \boldsymbol{v}_0^G$$

  - The angular momentum of a rigid body about A is equal to the sum of the angular momentum due to the rotational motion about the center of mass G ($\boldsymbol{H}_{G0}$), plus the angular momentum due to the translational motion of the center of mass as seen from A ($\boldsymbol{r}_A^G \times m\boldsymbol{v}_0^G$)

Kinetic energy

- We define the kinetic energy of a rigid body as the sum (integral) of the kinetic energies of all its constituent particles:

$$T_0 = \int_{\mathcal{D}} \mathrm{d}T_0 = \frac{1}{2}\int_{\mathcal{D}} (v_0^P)^2 \mathrm{d}m = \frac{1}{2}\int_{\mathcal{D}} (\boldsymbol{v}_0^{A(B)} + \boldsymbol{\omega}_{B0}\times \boldsymbol{r}_A^P)\cdot(\boldsymbol{v}_0^{A(B)} + \boldsymbol{\omega}_{B0}\times \boldsymbol{r}_A^P)\mathrm{d}m$$

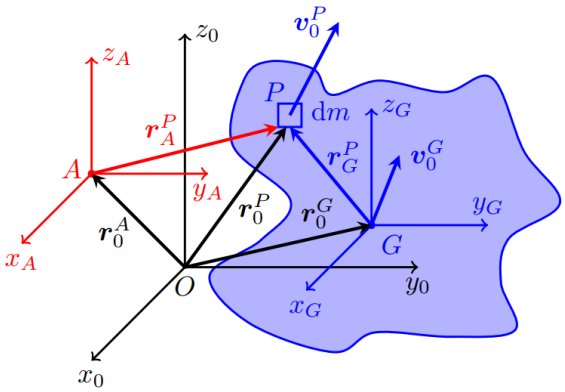
$$= \frac{1}{2} m (v_0^{A(B)})^2$$

$$\boldsymbol{v}_0^P = \boldsymbol{v}_0^{A(B)} + \boldsymbol{\omega}_{B0}\times \boldsymbol{r}_A^P$$

$$+ \left[\boldsymbol{\omega}_{B0}\times\left(\int_{\mathcal{D}} \boldsymbol{r}_A^P \mathrm{d}m\right)\right]\cdot \boldsymbol{v}_0^{A(B)} \qquad m\left(\boldsymbol{\omega}_{B0}\times \boldsymbol{r}_A^G\right)\cdot \boldsymbol{v}_0^{A(B)}$$

$$+ \frac{1}{2}\int_{\mathcal{D}} (\boldsymbol{\omega}_{B0}\times \boldsymbol{r}_A^P)\cdot(\boldsymbol{\omega}_{B0}\times \boldsymbol{r}_A^P)\mathrm{d}m$$

$$(\boldsymbol{A}\times\boldsymbol{B})\cdot(\boldsymbol{C}\times\boldsymbol{D}) = (\boldsymbol{A}\cdot\boldsymbol{C})(\boldsymbol{B}\cdot\boldsymbol{D}) - (\boldsymbol{A}\cdot\boldsymbol{D})(\boldsymbol{B}\cdot\boldsymbol{C})$$



$$(\boldsymbol{\omega}_{B0}\cdot\boldsymbol{\omega}_{B0})(\boldsymbol{r}_A^P\cdot\boldsymbol{r}_A^P) - (\boldsymbol{\omega}_{B0}\cdot\boldsymbol{r}_A^P)(\boldsymbol{r}_A^P\cdot\boldsymbol{\omega}_{B0})$$

$$\boldsymbol{\omega}_{B0}\cdot(r_A^P)^2\overline{\overline{\mathcal{U}}}\cdot\boldsymbol{\omega}_{B0} \qquad \boldsymbol{\omega}_{B0}\cdot(\boldsymbol{r}_A^P\otimes\boldsymbol{r}_A^P)\cdot\boldsymbol{\omega}_{B0}$$

$$\frac{1}{2}\int_{\mathcal{D}}\left[\boldsymbol{\omega}_{B0}\cdot(r_A^P)^2\overline{\overline{\mathcal{U}}}\cdot\boldsymbol{\omega}_{B0} - \boldsymbol{\omega}_{B0}\cdot(\boldsymbol{r}_A^P\otimes\boldsymbol{r}_A^P)\cdot\boldsymbol{\omega}_{B0}\right]\mathrm{d}m$$

$$\frac{1}{2}\boldsymbol{\omega}_{B0}\cdot\overline{\overline{\mathcal{I}}}_A\cdot\boldsymbol{\omega}_{B0}$$

Kinetic energy

- Gathering all terms:

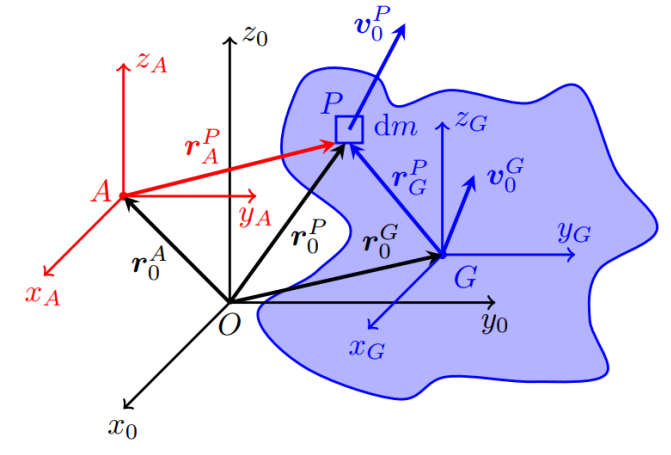
$$T_0 = \frac{1}{2}\boldsymbol{\omega}_{B0} \cdot \bar{\bar{\mathcal{I}}}_A \cdot \boldsymbol{\omega}_{B0} + \frac{1}{2} m (v_0^{A(B)})^2 + m \left(\boldsymbol{\omega}_{B0} \times \boldsymbol{r}_A^G\right) \cdot \boldsymbol{v}_0^{A(B)}$$

- Observe that this equation can be simplified if...
  - $A(B)$ is a body point that does not move, i.e., it is fixed in S_0:

$$T_0 = \frac{1}{2}\boldsymbol{\omega}_{B0} \cdot \bar{\bar{\mathcal{I}}}_A \cdot \boldsymbol{\omega}_{B0}$$

  - We choose the center of mass G as the reference point about which we compute the angular momentum:

$$T_0 = \frac{1}{2}\boldsymbol{\omega}_{B0} \cdot \bar{\bar{\mathcal{I}}}_G \cdot \boldsymbol{\omega}_{B0} + \frac{1}{2} m (v_0^G)^2$$



Kinetic energy

- A different expansion leads to a useful relation:

$$T_0 = \int_{\mathcal{D}} \mathrm{d}T_0 = \frac{1}{2}\int_{\mathcal{D}} (v_0^P)^2 \mathrm{d}m$$

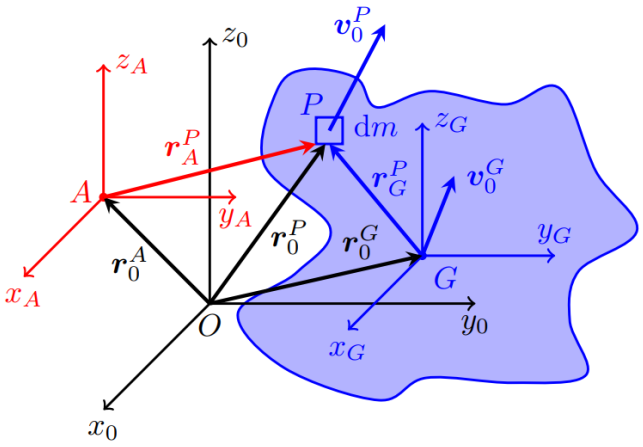
$$\boldsymbol{v}_0^P = \boldsymbol{v}_0^G + \boldsymbol{\omega}_{B0} \times \boldsymbol{r}_G^P$$

$$(v_0^P)^2 = \boldsymbol{v}_0^P \cdot \boldsymbol{v}_0^P = (v_0^G)^2 + \left|\boldsymbol{\omega}_{B0} \times \boldsymbol{r}_G^P\right|^2 + 2\boldsymbol{v}_0^G \cdot (\boldsymbol{\omega}_{B0} \times \boldsymbol{r}_G^P)$$

$$\frac{1}{2}\int_{\mathcal{D}} (v_0^G)^2 \mathrm{d}m = \frac{1}{2} m (v_0^G)^2$$

$$\int_{\mathcal{D}} \boldsymbol{v}_0^G \cdot (\boldsymbol{\omega}_{B0} \times \boldsymbol{r}_G^P)\mathrm{d}m = \boldsymbol{v}_0^G \cdot \left(\boldsymbol{\omega}_{B0} \times \int_{\mathcal{D}} \boldsymbol{r}_G^P \mathrm{d}m\right) = 0$$

$$\frac{1}{2}\int_{\mathcal{D}} \left|\boldsymbol{\omega}_{B0} \times \boldsymbol{r}_G^P\right|^2 \mathrm{d}m = T_G$$



- We find:

$$T_0 = T_G + \frac{1}{2} m (v_0^G)^2$$

  - The kinetic energy of the rigid body T_0 is equal to the sum of the kinetic energy of the body due to its rotational motion about the center of mass (T_G) plus the kinetic energy due to the translation of the center of mass ($m(v_0^G)^2/2$).

Summary of useful relations

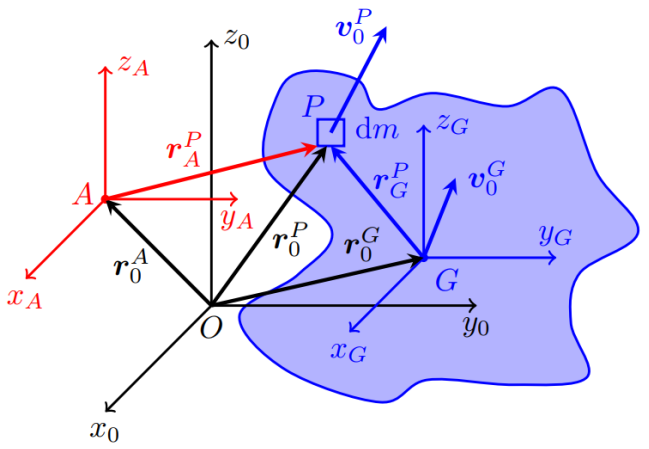
- Linear momentum

$$\boldsymbol{p}_0 = m\boldsymbol{v}_0^G$$

- Angular momentum

$$\boldsymbol{H}_{A,0} = \bar{\bar{\mathcal{I}}}_A \cdot \boldsymbol{\omega}_{B0} + m\boldsymbol{r}_A^G \times \boldsymbol{v}_0^{A(B)}$$

$$\boldsymbol{H}_{A,0} = \boldsymbol{H}_{G,0} + m\boldsymbol{r}_A^G \times \boldsymbol{v}_0^G \qquad \boldsymbol{H}_{G,G} \equiv \boldsymbol{H}_{G,0}$$



- Kinetic energy

$$T_0 = \frac{1}{2}\boldsymbol{\omega}_{B0} \cdot \bar{\bar{\mathcal{I}}}_A \cdot \boldsymbol{\omega}_{B0} + \frac{1}{2}m(v_0^{A(B)})^2 + m\left(\boldsymbol{\omega}_{B0} \times \boldsymbol{r}_A^G\right) \cdot \boldsymbol{v}_0^{A(B)}$$

$$T_0 = T_G + \frac{1}{2}m(v_0^G)^2$$

Dynamics of the rigid body

- We only have the equation of motion of a single point particle P_i.
  In an inertial reference frame S_0 we write:

$$m^{P_i}\boldsymbol{a}_0^{P_i} = \boldsymbol{F}_{P_i}$$

- We can think of a body as a collection of mass differentials and treat each of them as a point particle. Integrating over the whole body:

$$\int \boldsymbol{a}_0^P \,\mathrm{d}m = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 \int \boldsymbol{v}_0^P \,\mathrm{d}m = \int \mathrm{d}\boldsymbol{F}$$

- In the left-hand side we identify:

$$\left.\frac{\mathrm{d}\boldsymbol{p}_0}{\mathrm{d}t}\right|_0 = m^b \left.\frac{\mathrm{d}\boldsymbol{v}_0^G}{\mathrm{d}t}\right|_0 = m^b \boldsymbol{a}_0^G$$

  - The translational dynamics of a rigid body are simple: they are equivalent to the dynamics of a point particle with same mass, located and moving like the center of mass G, subjected to the effect of the *resultant of all forces acting on the body*

Dynamics of the rigid body

$$m^{P_i}\boldsymbol{a}_0^{P_i} = \boldsymbol{F}_{P_i}$$

- If we take the cross product with $\boldsymbol{AP_i}\times$

$$m^{P_i}\boldsymbol{AP_i}\times\boldsymbol{a}_0^{P_i} = \boldsymbol{AP_i}\times\boldsymbol{F}_{P_i} = \boldsymbol{M}_{A,P_i}$$

- But:

$$m^{P_i}\boldsymbol{AP_i}\times\boldsymbol{a}_0^{P_i} = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 \left(m^{P_i}\boldsymbol{AP_i}\times\boldsymbol{v}_0^{P_i}\right) + m^{P_i}\boldsymbol{v}_0^{A}\times\boldsymbol{v}_0^{P_i}$$

- For a differential mass particle, and integrating over the whole body:

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 \int \boldsymbol{AP}\times\boldsymbol{v}_0^{P}\mathrm{d}m + \boldsymbol{v}_0^{A}\times\int\boldsymbol{v}_0^{P}\mathrm{d}m = \int \mathrm{d}\boldsymbol{M}_A$$

- The left-hand side is actually:

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 \boldsymbol{H}_{A,0}^{b} + m^{b}\boldsymbol{v}_0^{A}\times\boldsymbol{v}_0^{G}$$

  - Typically we will take moments about a stationary point which is also a fixed point of the rigid body, or about the center of mass of the rigid body, to suppress the second term and simplify the expression of $\boldsymbol{H}_{A,0}^{b}$

Dynamics of the rigid body

$$m^{P_i}\boldsymbol{a}_0^{P_i} = \boldsymbol{F}_{P_i}$$

- If we take the dot product with $\boldsymbol{v}_0^{P_i}$

$$m^{P_i}\boldsymbol{v}_0^{P_i} \cdot \left.\frac{\mathrm{d}\boldsymbol{v}_0^{P_i}}{\mathrm{d}t}\right|_0 = \left.\frac{1}{2}m^{P_i}\frac{\mathrm{d}\left(v_0^{P_i}\right)^2}{\mathrm{d}t}\right|_0 = \boldsymbol{v}_0^{P_i} \cdot \boldsymbol{F}_{P_i} = \dot{W}_{P_i}$$

- For a differential mass particle, and integrating over the whole body:

$$\left.\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 \int \left(v_0^{P_i}\right)^2 \mathrm{d}m = \int \mathrm{d}\dot{W}$$

- The left-hand side is actually:

$$\left.\frac{\mathrm{d}T_0^b}{\mathrm{d}t}\right|_0$$

  - If the power of all forces acting on the body is zero or can be written as a potential term, then the mechanical (kinetic + potential) energy of the body is conserved in time

Thank you!

