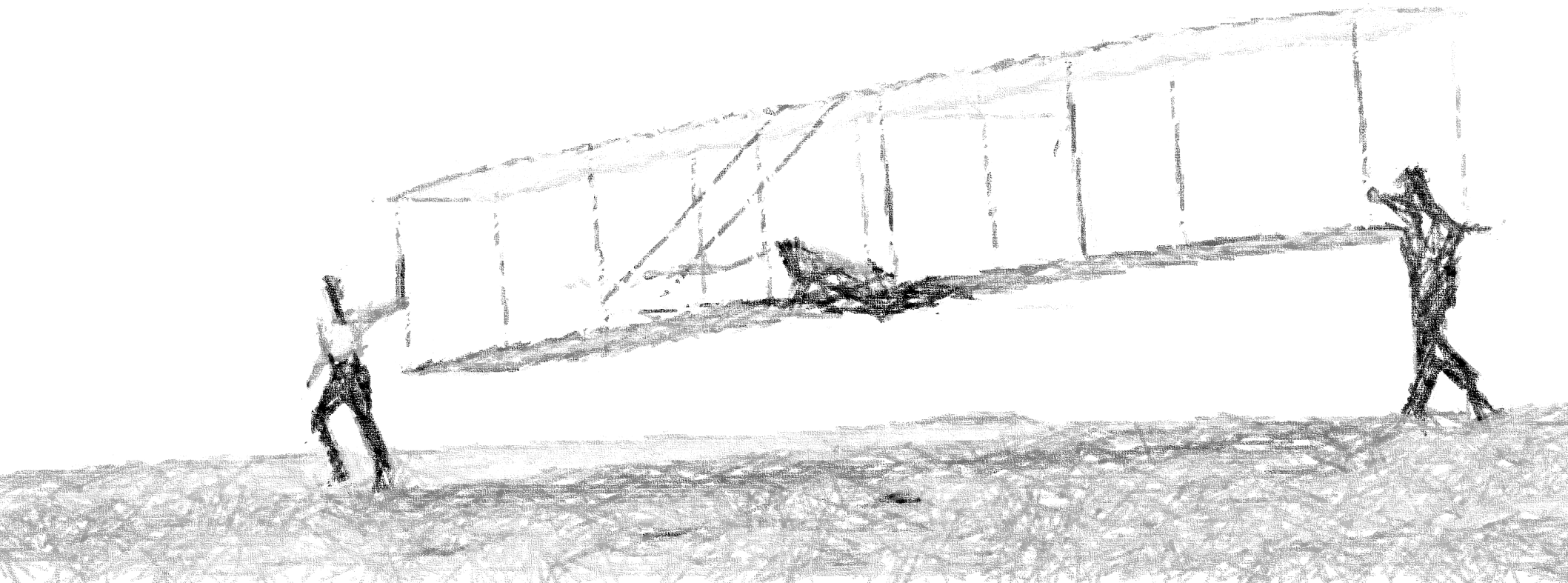
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Mechanics applied to Aerospace Engineering

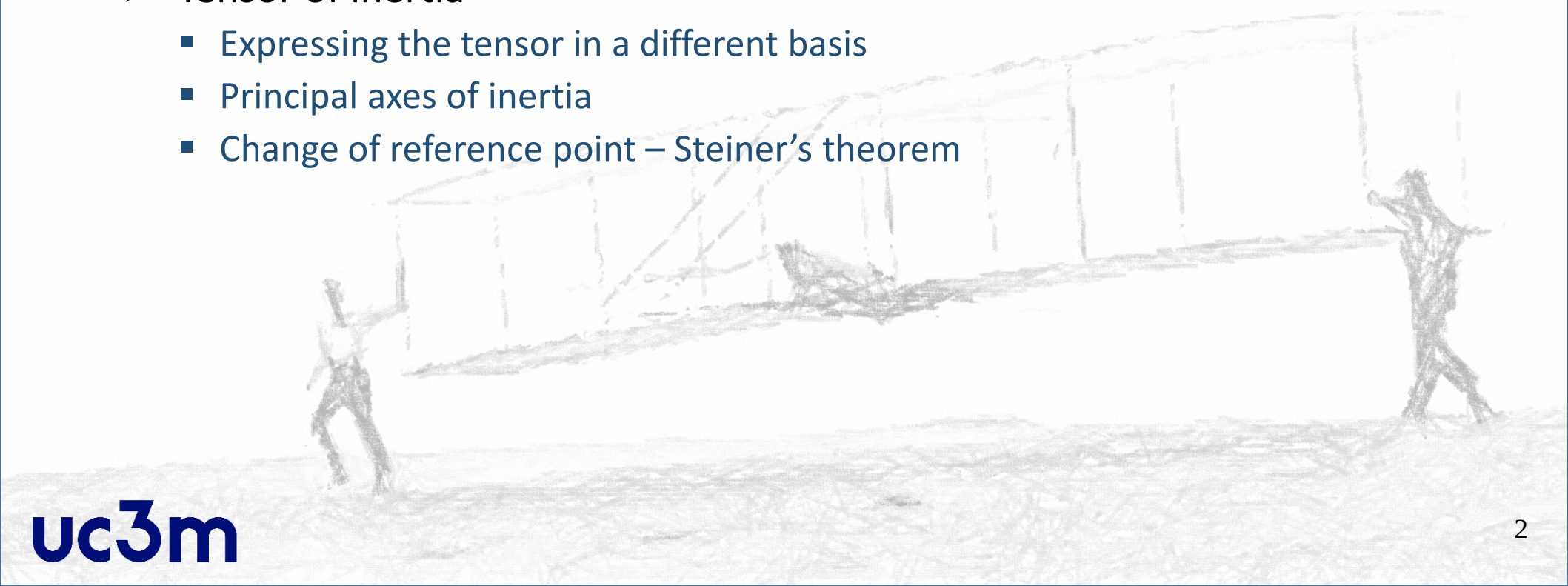
Course 251-14165 – BSc Aerospace Engineering

8. Inertia properties



Contents

- Tensors and tensor product
- The rigid body model and its inertia properties
- Moments of a mass distribution
- Total mass
- Center of mass
- Moments and products of inertia
- Tensor of inertia
  - Expressing the tensor in a different basis
  - Principal axes of inertia
  - Change of reference point – Steiner's theorem



Second order tensors

- A 2nd order tensor $\bar{\bar{T}}$ represents a linear relation among scalars, vectors, and other tensors. Here we are interested in second-order tensors only:
  - They can be dot-multiplied (*contracted*) with a vector, yielding a vector:

$$\bar{\bar{T}} \cdot \boldsymbol{b} = \boldsymbol{d}; \qquad \boldsymbol{a} \cdot \bar{\bar{T}} = \boldsymbol{c}$$

  - Or, dot-multiplied with a vector on each side, yielding a scalar:

$$\boldsymbol{a} \cdot \bar{\bar{T}} \cdot \boldsymbol{b} = s$$

- We can represent a vector in basis B_0 as a column matrix: $\boldsymbol{u} = \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix}_0$
  Similarly, a second-order tensor can be represented as a 2d matrix:

$$\bar{\bar{T}} = {}_0\begin{bmatrix} t_{xx} & t_{xy} & t_{xz} \\ t_{yx} & t_{yy} & t_{yz} \\ t_{zx} & t_{zy} & t_{zz} \end{bmatrix}_0$$

Unit tensor: $\bar{\bar{U}} = {}_0\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_0$

  - The operations above in matrix notation read:

$${}_0\begin{bmatrix} t_{xx} & t_{xy} & t_{xz} \\ t_{yx} & t_{yy} & t_{yz} \\ t_{zx} & t_{zy} & t_{zz} \end{bmatrix}_0 \cdot \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix}_0 = \begin{bmatrix} d_x \\ d_y \\ d_z \end{bmatrix}_0; \qquad \begin{bmatrix} a_x, a_y, a_z \end{bmatrix}_0 \cdot {}_0\begin{bmatrix} t_{xx} & t_{xy} & t_{xz} \\ t_{yx} & t_{yy} & t_{yz} \\ t_{zx} & t_{zy} & t_{zz} \end{bmatrix}_0 = \begin{bmatrix} c_x, c_y, c_z \end{bmatrix}$$

$$\begin{bmatrix} a_x, a_y, a_z \end{bmatrix}_0 \cdot {}_0\begin{bmatrix} t_{xx} & t_{xy} & t_{xz} \\ t_{yx} & t_{yy} & t_{yz} \\ t_{zx} & t_{zy} & t_{zz} \end{bmatrix}_0 \cdot \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix}_0 = s$$

Tensor product

- Two vectors $\boldsymbol{u} = \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix}_0$; $\boldsymbol{v} = \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}_0$ can be used to construct a tensor:

$$\bar{\bar{T}} = \boldsymbol{u} \otimes \boldsymbol{v} = \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix}_0 \cdot \begin{bmatrix} v_x, v_y, v_z \end{bmatrix}_0 = {}_0\begin{bmatrix} u_x v_x & u_x v_y & u_x v_z \\ u_y v_x & u_y v_y & u_y v_z \\ u_z v_x & u_z v_y & u_z v_z \end{bmatrix}_0$$

  - This operation is known as *tensor product*
  - Sometimes the symbol $\otimes$ is omitted and we just write; $\bar{\bar{T}} = \boldsymbol{uv}$ (that is why it is important to use the dot, when we mean dot product!)

- We can think of $\boldsymbol{u} \otimes \boldsymbol{v}$ as a special construct of these vectors that is "waiting" to be dot-multiplied with other vectors left and/or right:

$$\boldsymbol{a} \cdot (\boldsymbol{u} \otimes \boldsymbol{v}) = (\boldsymbol{a} \cdot \boldsymbol{u})\boldsymbol{v}; \quad (\boldsymbol{u} \otimes \boldsymbol{v}) \cdot \boldsymbol{b} = \boldsymbol{u}(\boldsymbol{v} \cdot \boldsymbol{b});$$

$$\boldsymbol{a} \cdot (\boldsymbol{u} \otimes \boldsymbol{v}) \cdot \boldsymbol{b} = (\boldsymbol{a} \cdot \boldsymbol{u})(\boldsymbol{v} \cdot \boldsymbol{b})$$

Expressing a tensor in a different basis

- Imagine we have a tensor $\overline{\overline{T}}$ expressed in basis B_0, and we want to express it in B_1. Assume we know $[{}_0R_1]$:

$$ {}_1\begin{bmatrix} t_{xx} & t_{xy} & t_{xz} \\ t_{yx} & t_{yy} & t_{yz} \\ t_{zx} & t_{zy} & t_{zz} \end{bmatrix}_1 = \left[{}_1R_0 \right] \cdot {}_0\begin{bmatrix} t_{xx} & t_{xy} & t_{xz} \\ t_{yx} & t_{yy} & t_{yz} \\ t_{zx} & t_{zy} & t_{zz} \end{bmatrix}_0 \cdot \left[{}_0R_1 \right] $$

$$ {}_1\begin{bmatrix} t_{xx} & t_{xy} & t_{xz} \\ t_{yx} & t_{yy} & t_{yz} \\ t_{zx} & t_{zy} & t_{zz} \end{bmatrix}_1 = \left[{}_0R_1 \right]^{-1} \cdot {}_0\begin{bmatrix} t_{xx} & t_{xy} & t_{xz} \\ t_{yx} & t_{yy} & t_{yz} \\ t_{zx} & t_{zy} & t_{zz} \end{bmatrix}_0 \cdot \left[{}_0R_1 \right] $$

$$ {}_1\left[\overline{\overline{T}}\right]_1 = \left[{}_0R_1 \right]^T \cdot {}_0\left[\overline{\overline{T}}\right]_0 \cdot \left[{}_0R_1 \right] $$

Symmetric tensors

- The most interesting tensors to us are *symmetric tensors*. The matrix in this case is symmetric:

$$\bar{\bar{T}} = \begin{matrix} \\ 0 \end{matrix} \begin{bmatrix} t_{xx} & t_{xy} & t_{xz} \\ t_{xy} & t_{yy} & t_{yz} \\ t_{xz} & t_{yz} & t_{zz} \end{bmatrix}_0$$

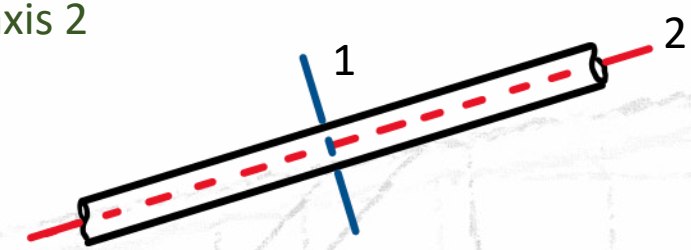
- For symmetric tensors, there always exists a vector basis of eigenvectors $B_* = \{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ (which we shall choose to be unitary). In addition, these vectors are always *orthogonal*. The diagonal components of the tensor in this new orthonormal basis are the associated *eigenvalues* $\lambda_1, \lambda_2, \lambda_3$*:*

$$\bar{\bar{T}} = \begin{matrix} \\ * \end{matrix} \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}_* $$

  - The directions given by $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$ are known as *principal directions* of the tensor

The rigid body model

- Remember the two differences of the rigid body model with respect to the point particle model:
  - It allows studying the *attitude* of the object
  - It retains the information of its *spatial mass distribution*
- The mass distribution D of a material system (e.g. a rigid body) describes how the system mass is located in space.
  - It can be continuous (i.e. a density) or discrete (i.e. several point particles)
  - It affects the rotational dynamics of the rigid body
    - E.g.: consider this metal bar. It takes more "effort " to set it into rotation about axis 1 than about axis 2



- Last lecture: we learned how to deal with the attitude of the rigid body: replacing it with a *body-fixed reference frame*
- This lecture: how to take into account the mass distribution of the rigid body in this body-fixed frame. We will introduce the ingredients that we will need in future lectures

Moments of a mass distribution

- Consider a mass distribution D. We define the zeroth, first, and second moments of the mass distribution about the origin O as:

$$\mathcal{M}^{\mathcal{D}}_{O,0} = \int_{\mathcal{D}} \mathrm{d}m; \quad \mathcal{M}^{\mathcal{D}}_{O,1} = \int_{\mathcal{D}} \boldsymbol{r}_0^P \mathrm{d}m; \quad \mathcal{M}^{\mathcal{D}}_{O,2} = \int_{\mathcal{D}} \boldsymbol{r}_0^P \otimes \boldsymbol{r}_0^P \mathrm{d}m.$$

- The mass differential can be written in terms of the density function $\rho(\boldsymbol{r}_0^P)$ and the volume differential as:

$$\mathrm{d}m = \rho \mathrm{d}V$$

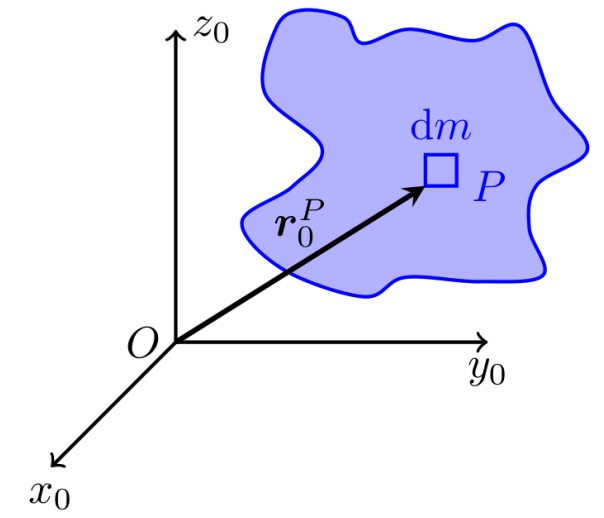
  - Likewise, for linear and planar mass distributions, with linear density $\lambda(\boldsymbol{r}_0^P)$ and surface density $\sigma(\boldsymbol{r}_0^P)$,

$$\mathrm{d}m = \lambda \mathrm{d}\ell; \qquad \mathrm{d}m = \sigma \mathrm{d}A$$

  - A mass distribution is homogeneous if ρ (or λ, σ) is uniform in space

- The volume differential can be expressed in different coordinate systems as:

$$\mathrm{d}V = \mathrm{d}x\mathrm{d}y\mathrm{d}z = R\mathrm{d}R\mathrm{d}\theta\mathrm{d}z = r^2 \cos\phi \mathrm{d}r\mathrm{d}\theta\mathrm{d}\phi$$



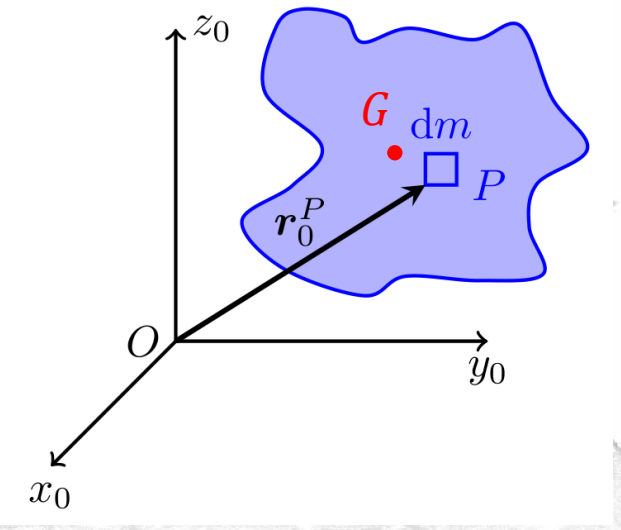
Total mass and center of mass

- The first inertia property of interest is the *total mass*. It is just the sum (i.e. the integral) of all the mass of the rigid body. It coincides with the zeroth moment of the mass distribution:

$$m = \int_{\mathcal{D}} \mathrm{d}m$$

- The *center of mass* is a special point of the rigid body. Its position is defined as the ratio (first moment)/(zeroth moment) of the mass distribution. It is a "weighted average" of the position of all mass differentials:

$$\boldsymbol{r}_0^G = \frac{1}{m}\int_{\mathcal{D}} \boldsymbol{r}_0^P \,\mathrm{d}m = \frac{\mathcal{M}^{\mathcal{D}}_{O,1}}{\mathcal{M}^{\mathcal{D}}_{O,0}}$$



Moment of inertia with respect to an axis

- Consider a rigid body b and an axis r. Calling d the distance of each point in the body to the axis, we define the moment of inertia of b with respect to r as:

$$I_r^b = \int_{\mathcal{D}} d^2 \mathrm{d}m$$

  - I_r^b is the "resistance" the body offers to changes in its angular velocity about this axis
  - If the axis r is one of the coordinated axes (Ox, Oy or Oz) we may write:

$$I_x = \int_{\mathcal{D}} (y^2 + z^2)\, \mathrm{d}m; \quad I_y = \int_{\mathcal{D}} (x^2 + z^2)\, \mathrm{d}m; \quad I_z = \int_{\mathcal{D}} (x^2 + y^2)\, \mathrm{d}m$$

- For completeness, we can also define a moment of inertia with respect to a point (e.g. O) or with respect to a plane (e.g. Oxy)

$$I_O = \int_{\mathcal{D}} (x^2 + y^2 + z^2)\, \mathrm{d}m \qquad I_{xy} = \int_{\mathcal{D}} z^2 \mathrm{d}m$$

- A useful relation:

$$2I_O = 2\,(I_{xy} + I_{xz} + I_{yz}) = I_x + I_y + I_z$$

Products of inertia

- Another quantity that we must define is the *product of inertia* with respect to two orthogonal planes π_1, π_2. Calling d_1, d_2 the *signed* distances of each point of the rigid body to each plane,

$$P_{\pi_1,\pi_2} = \int_{\mathcal{D}} d_1 d_2 \mathrm{d}m$$

  - If the planes are the coordinated planes (Oxy, Oxz and Oyz) we may write

$$P_{xy,xz} = \int_{\mathcal{D}} zy\mathrm{d}m; \quad P_{xy,yz} = \int_{\mathcal{D}} zx\mathrm{d}m$$

$$P_{xz,yz} = \int_{\mathcal{D}} yx\mathrm{d}m$$

  - The "meaning" of $P_{xy,xz}$ is to measure the "coupling" that exists between the rotation about the Oy axis and the Oz axis

If the mass distribution is symmetric about one of the involved planes, then the product of inertia is zero

Tensor of inertia

- The object that describes the rotational inertia of a rigid body is the *tensor of inertia* about a point *O*:

$$\bar{\bar{\mathcal{I}}}_0 = \int_{\mathcal{D}} \left[(r_0^P)^2 \bar{\bar{\mathcal{U}}} - \boldsymbol{r}_0^P \otimes \boldsymbol{r}_0^P \right] \mathrm{d}m$$

where:

$$(r_0^P)^2 \mathcal{U} = {}_0\begin{bmatrix} x^2+y^2+z^2 & 0 & 0 \\ 0 & x^2+y^2+z^2 & 0 \\ 0 & 0 & x^2+y^2+z^2 \end{bmatrix}_0 \qquad \boldsymbol{r}_0^P \otimes \boldsymbol{r}_0^P = {}_0\begin{bmatrix} x^2 & xy & xz \\ xy & y^2 & yz \\ xz & yz & z^2 \end{bmatrix}_0$$

Therefore

$$\bar{\bar{\mathcal{I}}}_0 = \int_{\mathcal{D}} {}_0\begin{bmatrix} y^2+z^2 & -xy & -xz \\ -xy & x^2+z^2 & -yz \\ -xz & -yz & x^2+y^2 \end{bmatrix}_0 \mathrm{d}m$$

$$= {}_0\begin{bmatrix} I_x & -P_{xz,yz} & -P_{xy,yz} \\ -P_{xz,yz} & I_y & -P_{xy,xz} \\ -P_{xy,yz} & -P_{xy,xz} & I_z \end{bmatrix}_0$$

The components of tensor $\bar{\bar{I}}_O$ in a vector basis B_0 are the moments and products of inertia with respect to the coordinated axes and planes

It is a *symmetric*, definite-positive tensor

$\boldsymbol{u} \cdot \bar{\bar{I}}_O \cdot \boldsymbol{u} > 0 \;\; \forall \boldsymbol{u}: u > 0$

Tensor of inertia

$$\bar{\bar{\mathcal{I}}}_0 = {}_0\begin{bmatrix} I_x & -P_{xz,yz} & -P_{xy,yz} \\ -P_{xz,yz} & I_y & -P_{xy,xz} \\ -P_{xy,yz} & -P_{xy,xz} & I_z \end{bmatrix}_0$$

- To transform the tensor components to a different basis:

$$ {}_1\left[\bar{\bar{I}}_O\right]_1 = \left[{}_0R_1\right]^T \cdot {}_0[\bar{\bar{I}}_O]_0 \cdot \left[{}_0R_1\right]$$

- The moment of inertia about an axis r passing through O and direction given by unit vector $\boldsymbol{e}$ can be computed as:

$$I_r = \boldsymbol{e} \cdot \bar{\bar{I}}_O \cdot \boldsymbol{e}$$

- If the mass distribution is symmetric about at least two coordinated planes, the tensor of inertia is *diagonal*

$$\bar{\bar{\mathcal{I}}}_0 = {}_1\begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix}_1$$

- Any orthonormal vector basis in which I_O is diagonal is called a *principal basis* of inertia
  - The vectors of the basis are eigenvectors of the tensor and are called the *principal directions of inertia*
  - The components in the diagonal are eigenvalues of the tensor and are called the *principal moments of inertia* of the body

To find the principal directions and moments we must *diagonalize* the tensor. Identifying any symmetries in the mass distribution helps

Steiner's theorem

- How do we compute the tensor of inertia about a different point?
- Consider a body b with center of mass G and a point O. We want to find the relation between tensors $\overline{\overline{I}}_O$ and $\overline{\overline{I}}_G$:

$$\overline{\overline{\mathcal{I}}}_0 = \int_{\mathcal{D}} \left[(r_0^P)^2 \overline{\overline{\mathcal{U}}} - \boldsymbol{r}_0^P \otimes \boldsymbol{r}_0^P \right] \mathrm{d}m \qquad \overline{\overline{\mathcal{I}}}_G = \int_{\mathcal{D}} \left[(r_G^P)^2 \overline{\overline{\mathcal{U}}} - \boldsymbol{r}_G^P \otimes \boldsymbol{r}_G^P \right] \mathrm{d}m$$

- We write $\boldsymbol{r}_0^P = \boldsymbol{r}_0^G + \boldsymbol{r}_G^P$ (i.e., $\boldsymbol{OP} = \boldsymbol{OG} + \boldsymbol{GP}$) and expand $\overline{\overline{I}}_O$:

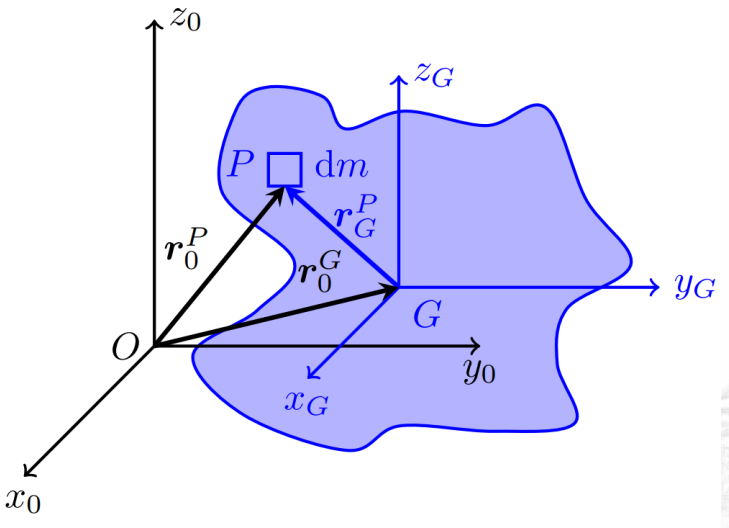
$$(r_0^P)^2 \overline{\overline{U}} = \left(r_0^G\right)^2 \overline{\overline{U}} + \left(r_G^P\right)^2 \overline{\overline{U}} + 2\boldsymbol{r}_0^G \cdot \boldsymbol{r}_G^P \overline{\overline{U}}$$

$$\boldsymbol{r}_0^P \otimes \boldsymbol{r}_0^P = \boldsymbol{r}_0^G \otimes \boldsymbol{r}_0^G + \boldsymbol{r}_G^P \otimes \boldsymbol{r}_G^P + \boldsymbol{r}_0^G \otimes \boldsymbol{r}_G^P + \boldsymbol{r}_G^P \otimes \boldsymbol{r}_0^G$$

$$\int_{\mathcal{D}} \boldsymbol{r}_G^P \mathrm{d}m = \boldsymbol{0}$$

$$\int_{\mathcal{D}} (\boldsymbol{r}_0^G \cdot \boldsymbol{r}_G^P) \overline{\overline{\mathcal{U}}} \mathrm{d}m = \boldsymbol{r}_0^G \cdot \left(\int_{\mathcal{D}} \boldsymbol{r}_G^P \mathrm{d}m \right) \overline{\overline{\mathcal{U}}} = \overline{\overline{0}}$$

$$\int_{\mathcal{D}} (\boldsymbol{r}_G^P \otimes \boldsymbol{r}_0^G) \mathrm{d}m = \left(\int_{\mathcal{D}} \boldsymbol{r}_G^P \mathrm{d}m \right) \otimes \boldsymbol{r}_0^G = \overline{\overline{0}}$$



Steiner's theorem

- After gathering terms we find:

$$\overline{\overline{\mathcal{I}}}_0 = \overline{\overline{\mathcal{I}}}_G + m\left[(r_0^G)^2\overline{\overline{\mathcal{U}}} - \boldsymbol{r}_0^G \otimes \boldsymbol{r}_0^G\right]$$

  - The tensor of inertia of a rigid body about a point O is equal to the tensor of inertia about the center of mass G, plus an additional contribution that is equal to the tensor of inertia about O of a point particle with the same mass of the rigid body located at G

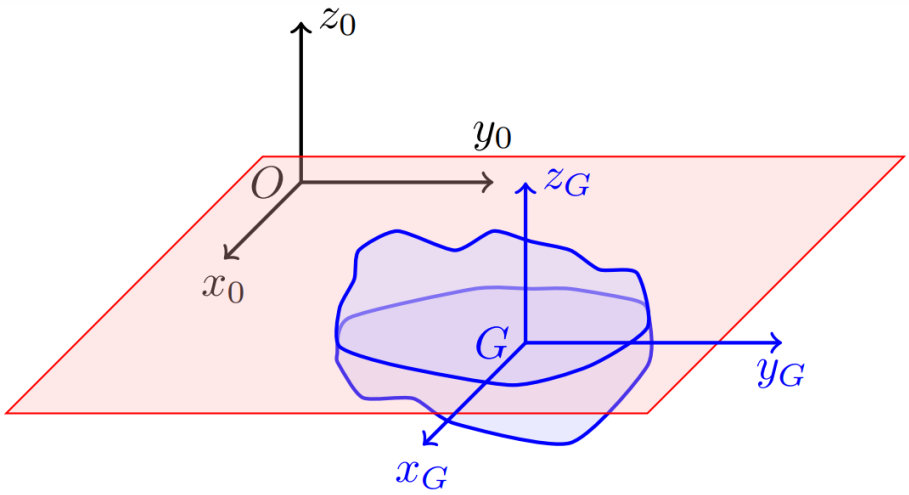
- This expression is useful to relate the tensor of inertia about an arbitrary point A to the tensor of inertia about the center of mass G

Caution! if you want to relate the tensor of inertia about a point A and a point B, and neither is the center of mass, you have to apply Steiner's theorem twice, to first go from A to G, and then from G to B.

Remember to compute all vector and tensor operations in the same basis. You may need to rotate (i.e. change basis) some vectors/tensors.

Parallel axis theorem

- This is just a particularization of Steiner's theorem
- Consider that we know the moment of inertia of a body b about an axis Gz_G, and we want to compute the moment of inertia about a parallel axis, Oz_0.



- Using Steiner's theorem and focusing only on the I_z component of the tensors:

$$I_{z_0} = I_{z_G} + m\left[(x_0^G)^2 + (y_0^G)^2\right]$$

  - This version of the theorem is useful in planar problems, where we will not be needing the full tensor, just the moment of inertia perpendicular to the plane of motion

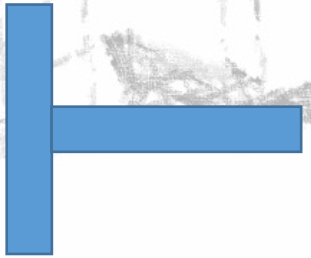
I_{zG} is always smaller than I_{z0}, regardless of where is O

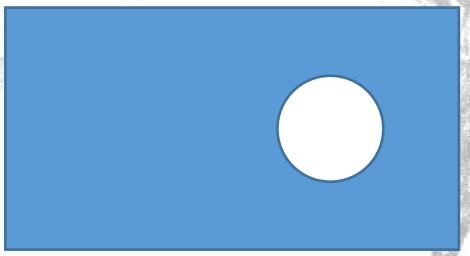
Hints

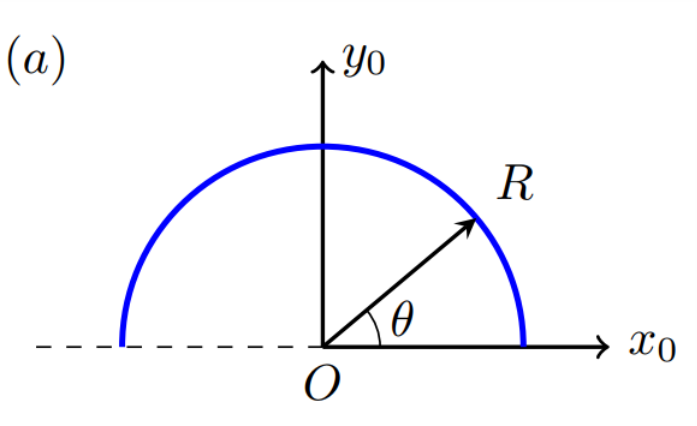
- To compute the mass, center of mass, and tensor of inertia of a rigid body, we need to solve triple integrals. Hints:
  - Use any symmetries in the mass distribution to simplify the integral
    - This is particularly useful to assert whether a product of inertia is zero or not
  - Divide and conquer: break down complex geometries into simpler ones and add their contributions
  - "Holes" in the mass distribution can be dealt with by subtraction
  - When it comes down to solving the integrals, express the volume differential in terms of a chosen set of coordinates:

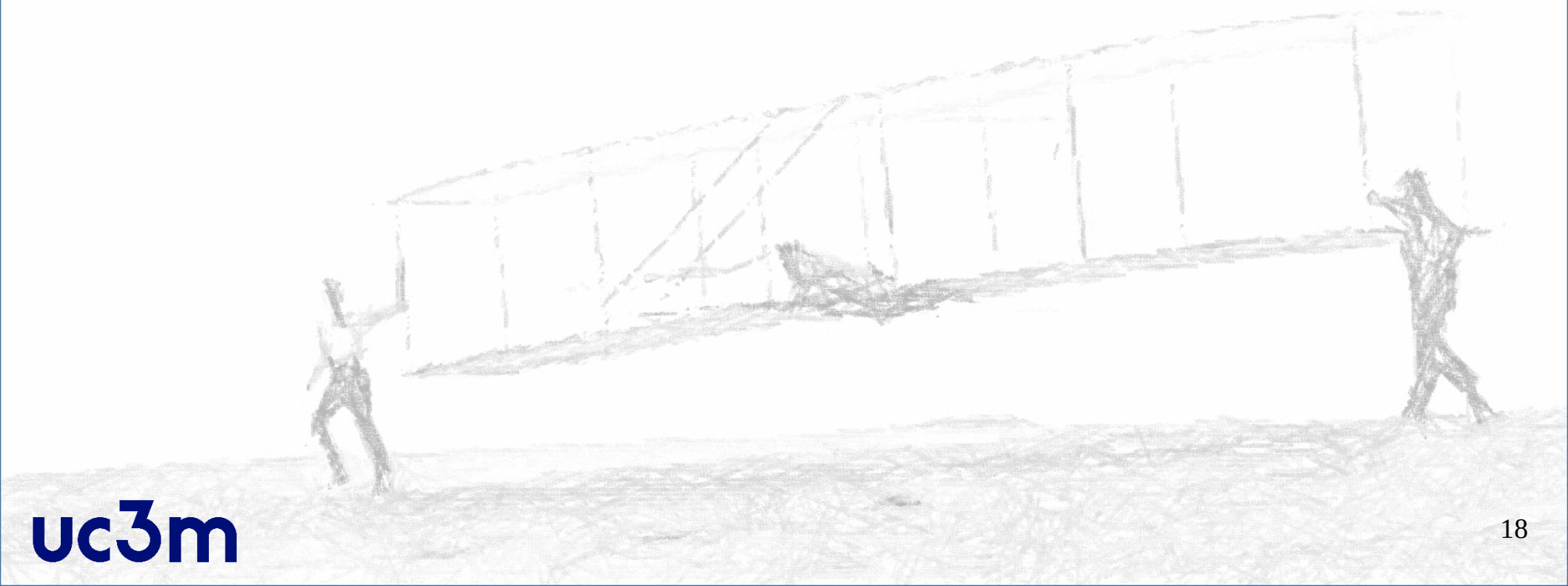
$$\mathrm{d}V = \mathrm{d}x\mathrm{d}y\mathrm{d}z = R\mathrm{d}R\mathrm{d}\theta\mathrm{d}z = r^2\cos\phi\mathrm{d}r\mathrm{d}\theta\mathrm{d}\phi$$

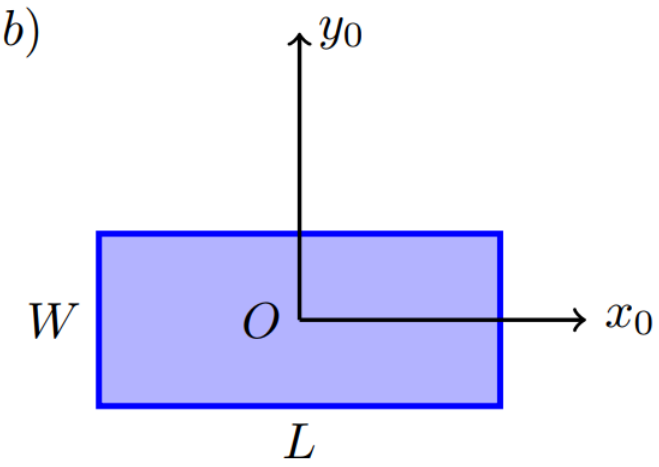


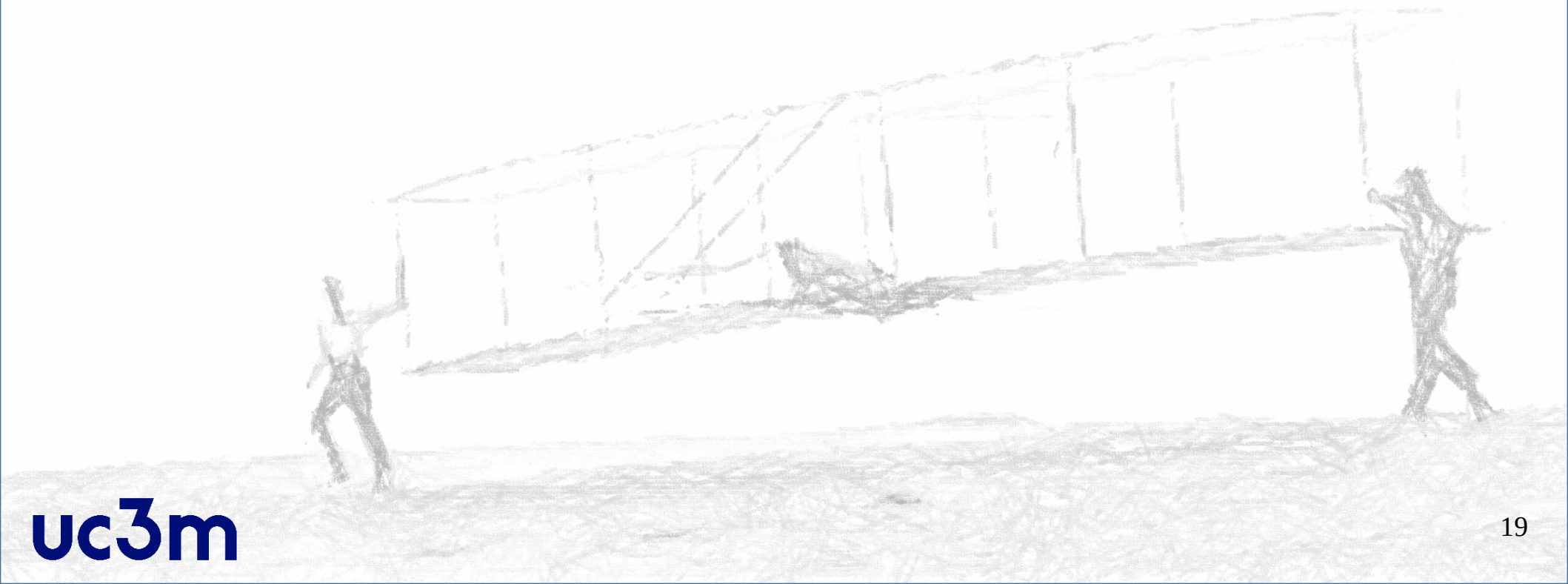




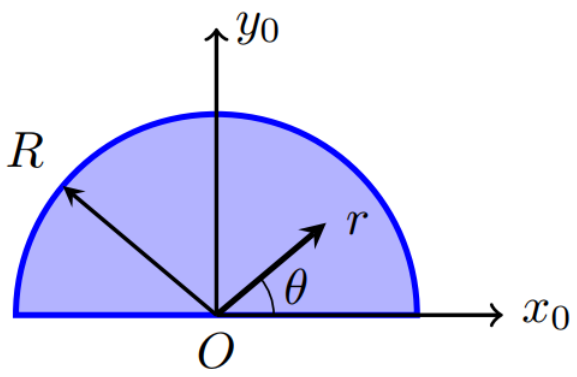

(a)
y_0
R
θ
x_0
O

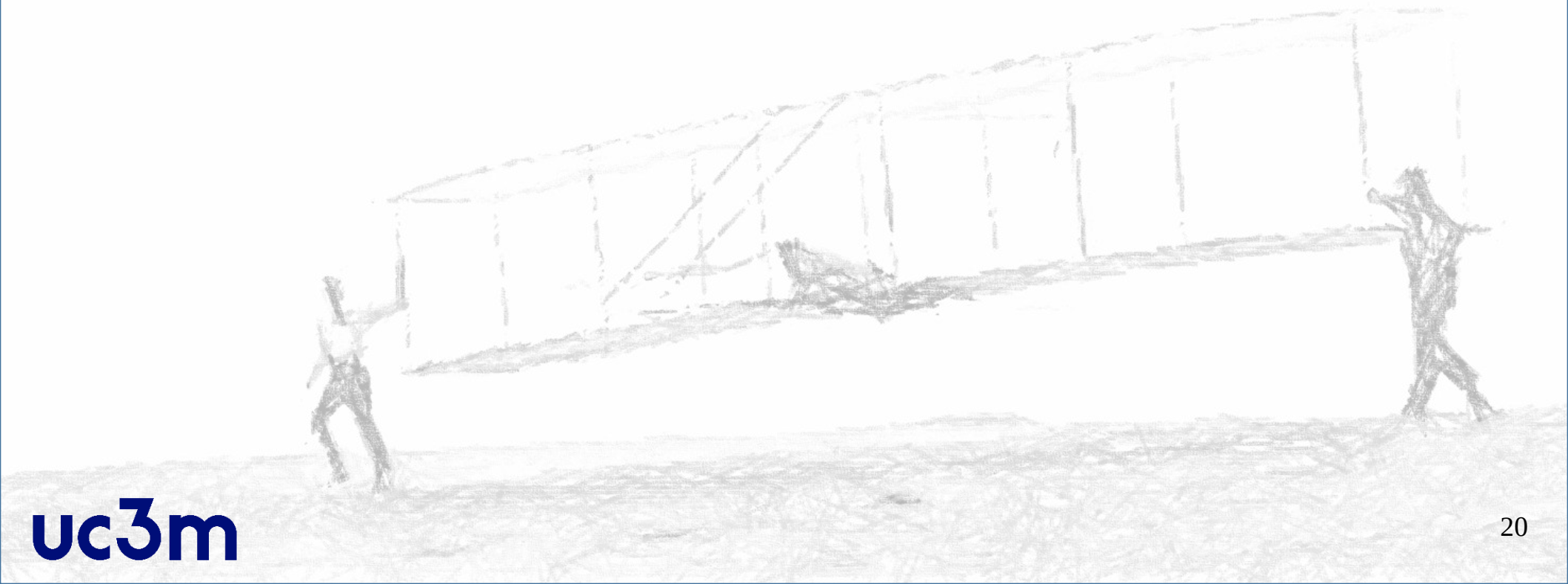


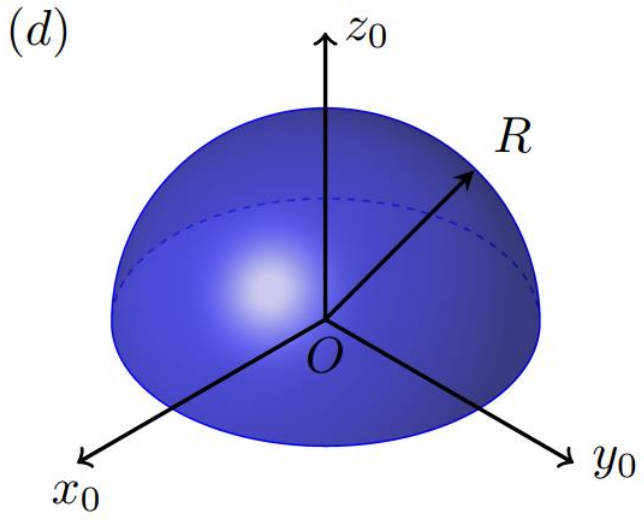

(b)
y_0
W
O
x_0
L

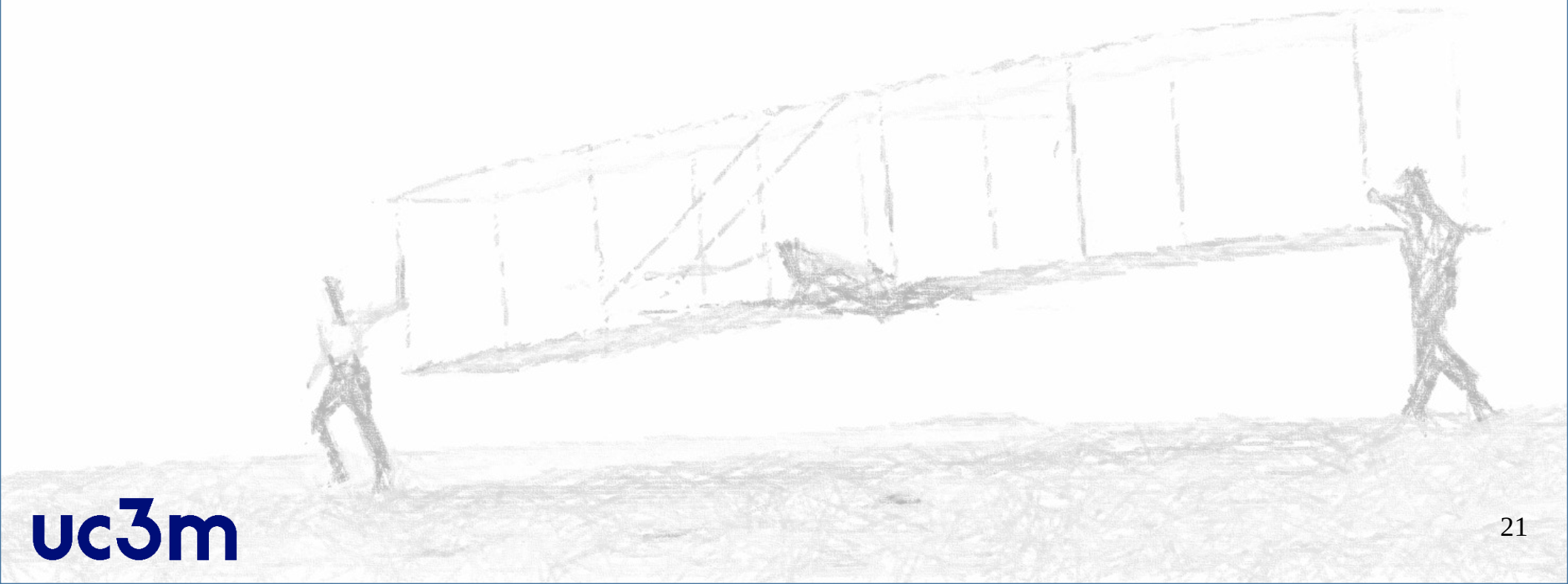


(c)






(d)
z_0
R
O
y_0
x_0



Thank you!

