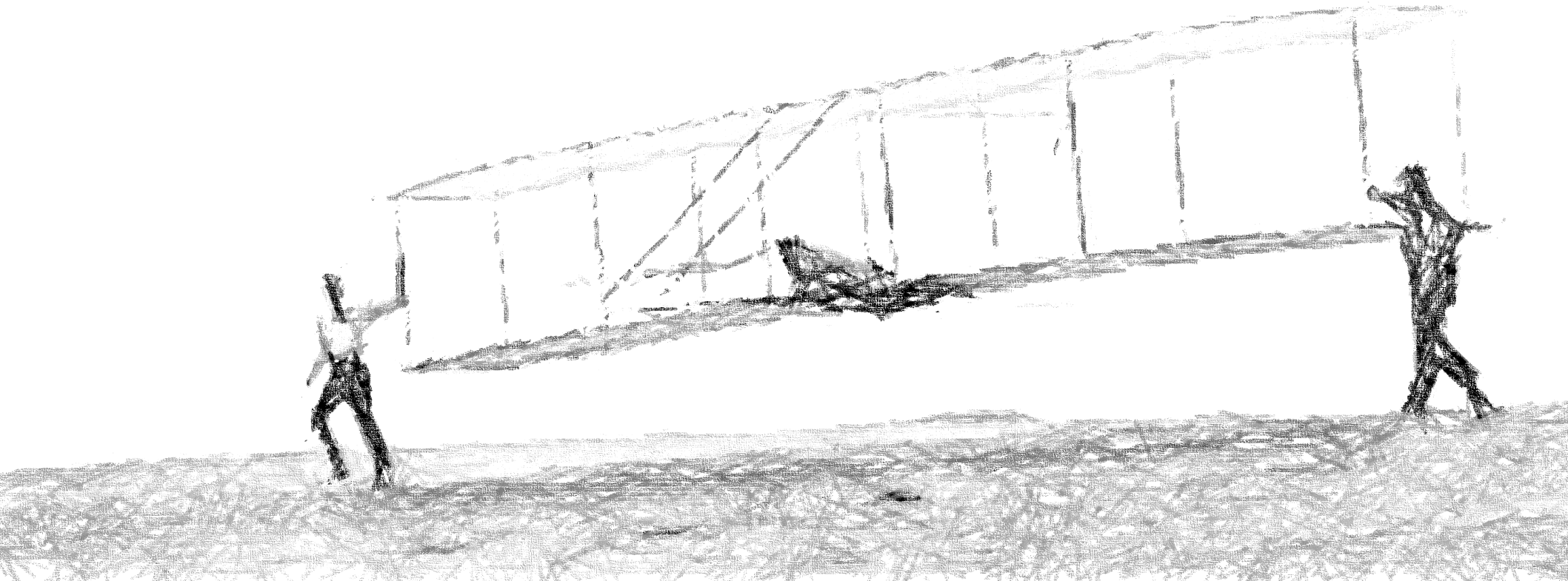
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Mechanics applied to Aerospace Engineering

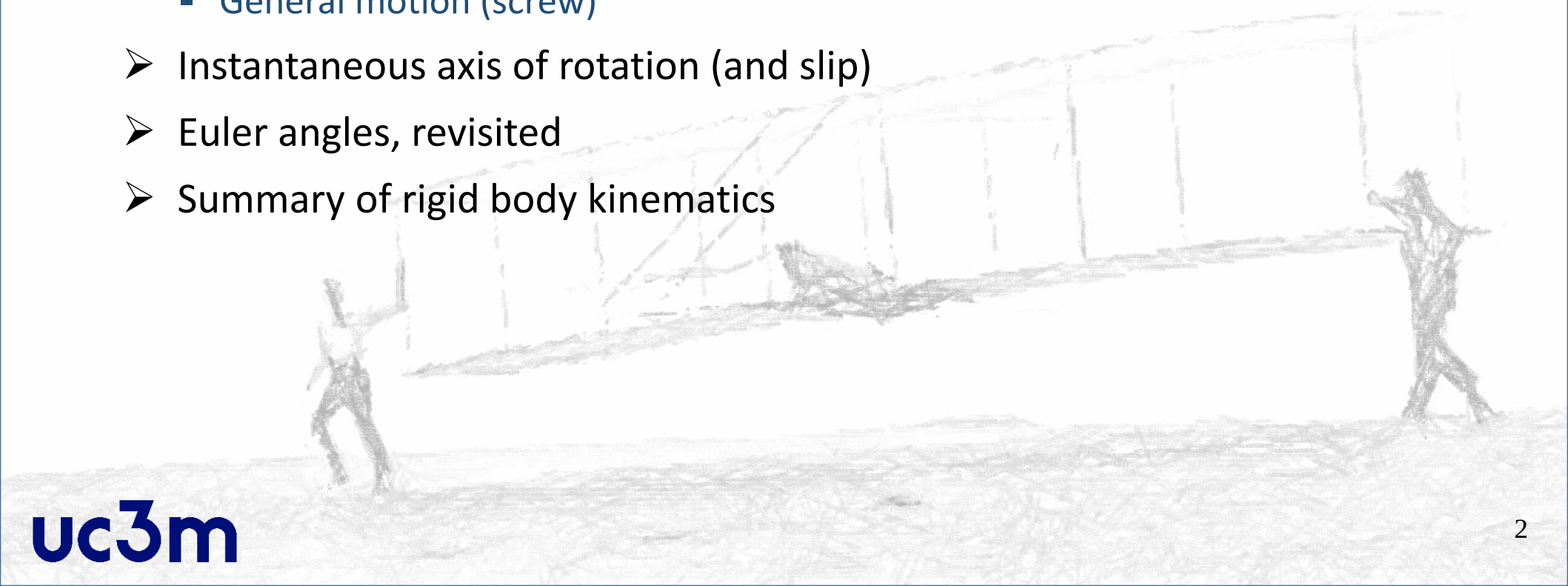
Course 251-14165 – BSc Aerospace Engineering

7. Rigid body kinematics



Contents

- The rigid body model
- Body-fixed reference frames
  - Rigid body – reference frame equivalence
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- Velocity and acceleration field of a rigid body
  - Pure translation
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  - General motion (screw)
- Instantaneous axis of rotation (and slip)
- Euler angles, revisited
- Summary of rigid body kinematics



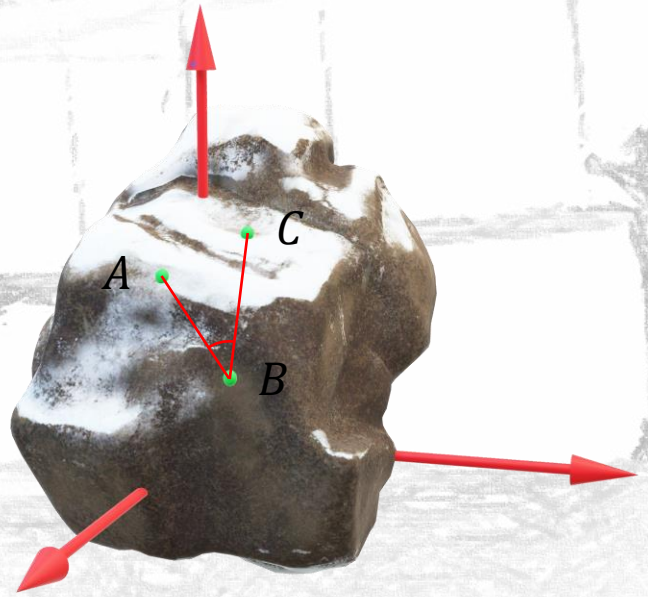
The rigid body model

- The point particle model has limitations: it ignores the *attitude* of the object and its *spatial mass distribution*
- The rigid body model allows taking these two aspects into account
- We say a body is *rigid* if the following kinematic constraint is satisfied:

"the distance between any two body points A and B remains constant in time"

  - As a consequence, the angle ABC between body points A, B, C is constant
  - That is, the body is perfectly indeformable and all its points move rigidly
  - More advanced models (deformable bodies) exist, but they are outside the scope of this course

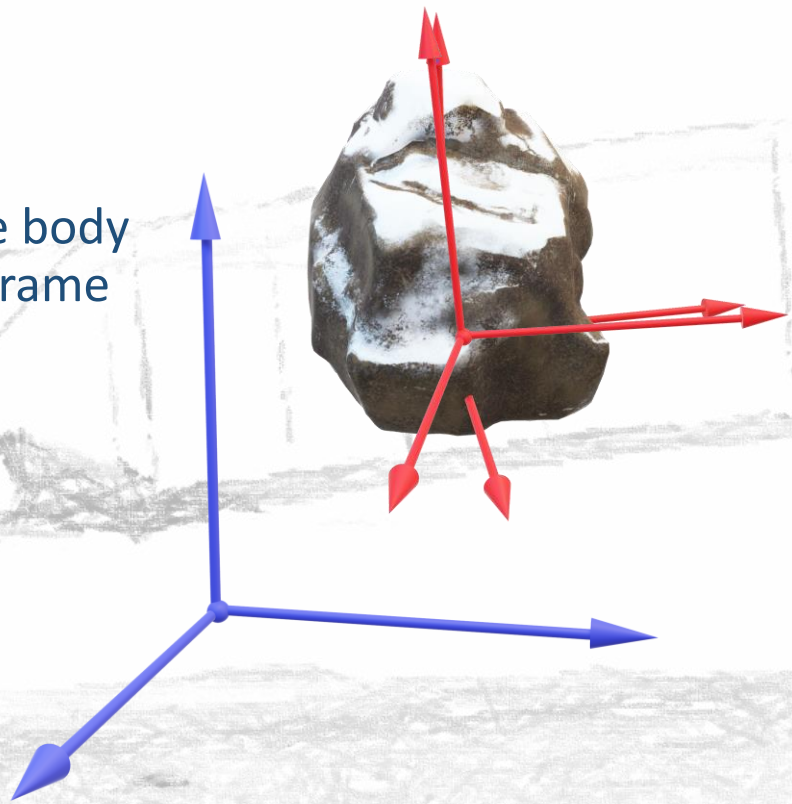
- We can think of a rigid body as a collection of N point particles.
  - To describe their configuration, we would need $3N$ coordinates!
  - However, the rigid body constraint reduces the number of degrees of freedom of a free rigid body to only 6



Body-fixed reference frames

- We can define a *body-fixed reference frame* S_b as follows:
  - Pick any body point A as the origin
  - Pick two other body points B and C (not colinear with A) and define the vectors $\boldsymbol{AB}$, $\boldsymbol{AC}$, $\boldsymbol{AB} \times \boldsymbol{AC}$ as the vectors of the basis (they don't even need to be orthonormal and right-handed—but we can choose them to be)
- Observe that:
  - Frame S_b translates and rotates with the body
  - All body points are stationary in S_b (they have zero relative velocity and acceleration)
  - We can describe the attitude of the body with the attitude of the reference frame
  - The definition of S_b is not unique

From the kinematics viewpoint, a rigid body is fully equivalent to a reference frame. We can ignore the particular shape and dimensions of the body and replace it with S_b



Rigid body points

- What is a *point of the rigid body*? Any point P that travels with the body-fixed reference frame S_b and therefore has $\boldsymbol{v}_b^P = \mathbf{0}$ and $\boldsymbol{a}_b^P = \mathbf{0}$ all the time.
  - A body point can be a material point (part of the physical body) or even a point outside of it!
- In case of ambiguity, we will use the following notation to specify that we are talking of a point A that travels with S_b and is a body point:

$$\boldsymbol{v}_0^{\mathrm{A(b)}}$$

  - This clarification is only important when we want to distinguish a geometric point of space A from the body point that happens to be on top of it at a particular instant of time. Note that

$$\boldsymbol{v}_0^A \neq \boldsymbol{v}_0^{A(b)}$$

since $\boldsymbol{v}_b^{A(b)} = \mathbf{0}$ but $\boldsymbol{v}_b^A \neq \mathbf{0}$ in general.

Velocity and acceleration fields

- The rigid body constraint means $\boldsymbol{v}_b^P = \boldsymbol{0}$ and $\boldsymbol{a}_b^P = \boldsymbol{0}$ $\forall P \in b$. Hence:

$$\boldsymbol{v}_0^{\mathrm{P}} = \cancel{\boldsymbol{v}_0^{\mathrm{P}}} = \boldsymbol{v}_0^{\mathrm{A}} + \boldsymbol{\omega}_{b0} \times \boldsymbol{AP},$$

$$\boldsymbol{a}_0^{\mathrm{P}} = \cancel{\boldsymbol{a}_0^{\mathrm{P}}} = \boldsymbol{a}_0^{\mathrm{A}} + \boldsymbol{\alpha}_{b0} \times \boldsymbol{AP} + \boldsymbol{\omega}_{b0} \times (\boldsymbol{\omega}_{b0} \times \boldsymbol{AP}) \cancel{+ \boldsymbol{v}_b^{\mathrm{P}}}.$$

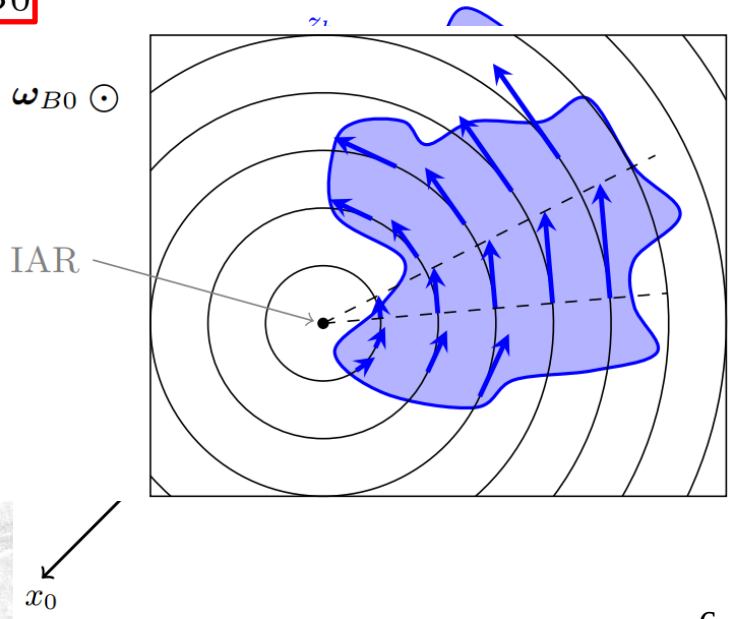
- The instantaneous velocity of any body point P can be written as the velocity of a chosen reference point A plus a rotation about A with $\boldsymbol{\omega}_{b0}$
  - If $\boldsymbol{\omega}_{b0} = \boldsymbol{0}$, $\boldsymbol{v}_0^P = \boldsymbol{v}_0^A$ $\forall P$, this is a *pure translation* at that instant
- Observe that the component of the velocity of all body points along $\boldsymbol{\omega}_{b0}$ is constant: we call this the *slip speed:*

$$\boldsymbol{v}_0^{\mathrm{P}} \cdot \boldsymbol{e}_\omega = \boldsymbol{v}_0^{\mathrm{A}} \cdot \boldsymbol{e}_\omega = v_{0\parallel} \qquad \boldsymbol{e}_\omega = \boldsymbol{\omega}_{b0}/\omega_{B0}$$

  - The velocity of any point P can be expressed as the sum of the body's parallel slip velocity $\boldsymbol{v}_{0\parallel} = v_{0\parallel}\boldsymbol{e}_\omega$ plus the particular perpendicular velocity:

$$\boldsymbol{v}_0^{\mathrm{P}} = \boldsymbol{v}_{0\parallel} + \boldsymbol{v}_{0\perp}^P$$

  - If $v_{0\parallel} = 0$, the motion is a *pure rotation* at that instant. The points with zero velocity are the *instantaneous axis of rotation*, parallel to $\boldsymbol{\omega}_{b0}$
    - This axis may be *outside* the material body



Velocity and acceleration fields

$$\boldsymbol{v}_0^{\mathrm{P}} = \boldsymbol{v}_0^{\mathrm{A}} + \boldsymbol{\omega}_{b0} \times \boldsymbol{AP}$$

- Also, the components of the velocities of any two points A and P along $\boldsymbol{AP}$ are equal:

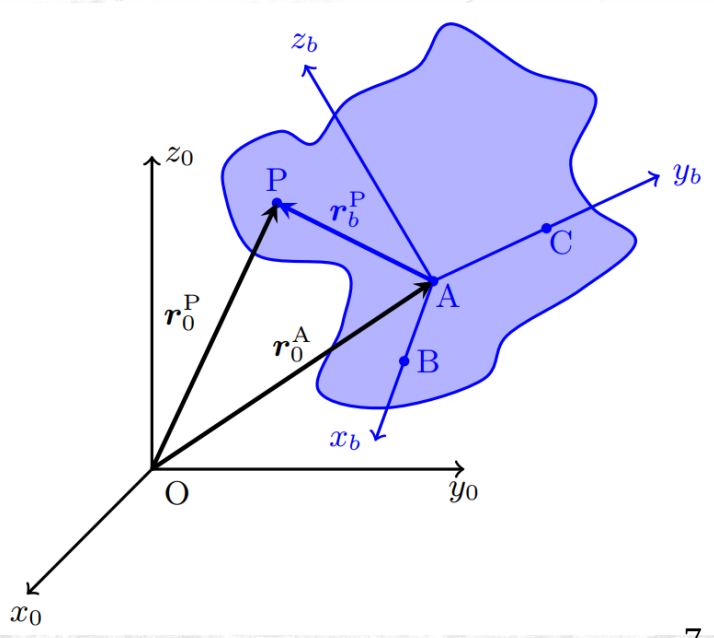
$$\boldsymbol{v}_0^{\mathrm{P}} \cdot \boldsymbol{AP} = \boldsymbol{v}_0^{\mathrm{A}} \cdot \boldsymbol{AP}$$

- If we know the velocity of three non-colinear points A, B, C we can use the expression above to find $\boldsymbol{\omega}_{b0}$:

$$\boldsymbol{v}_0^{B} = \boldsymbol{v}_0^{\mathrm{A}} + \boldsymbol{\omega}_{b0} \times \boldsymbol{AB}$$
$$\boldsymbol{v}_0^{C} = \boldsymbol{v}_0^{\mathrm{A}} + \boldsymbol{\omega}_{b0} \times \boldsymbol{AC}$$

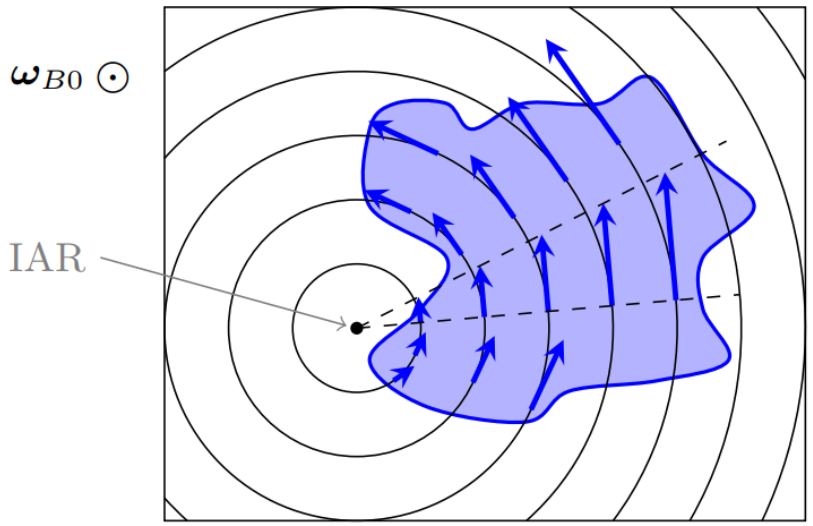
  - The data need to be *compatible*!
    Else, the object is not a rigid body

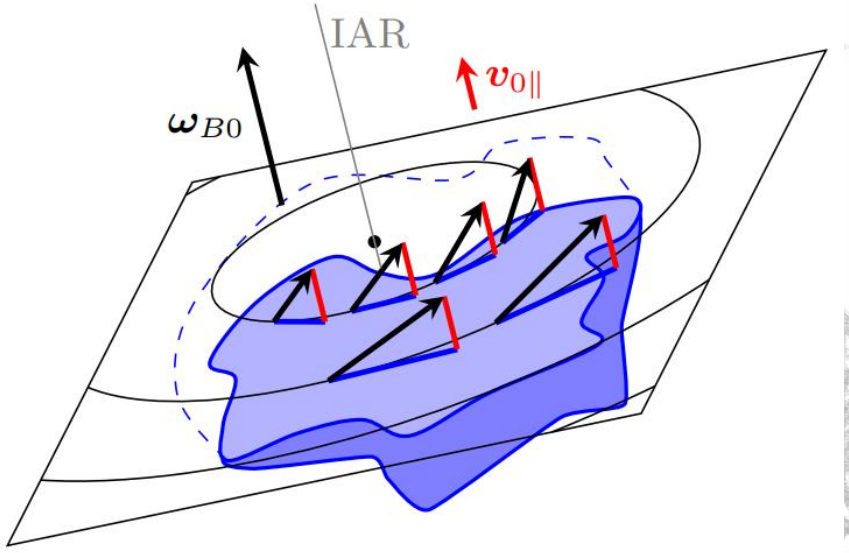
Completely determining the *instantaneous* velocity field of a rigid body is equivalent to knowing $\boldsymbol{\omega}_{b0}$ and the velocity of a single point A, $\boldsymbol{v}_0^A$



Instantaneous axis of rotation (and slip)

- In the most general case, the motion of S_b can be decomposed into a pure rotation plus a pure translation parallel to $\boldsymbol{\omega}_{b0}$ with slip velocity $\boldsymbol{v}_{0\parallel}$. We call this motion *screw motion*
- The *instantaneous axis of rotation and slip* is the locus of points with *minimum* velocity of the rigid body.
  - This velocity is $\boldsymbol{v}_{0\parallel}$
  - The axis direction is $\boldsymbol{\omega}_{b0}$





Instantaneous axis of rotation (and slip)

- To find a point I of the instantaneous axis of rotation and slip, we impose that its velocity be $\boldsymbol{v}_{0\parallel}$ and cross-multiply by $\boldsymbol{\omega}_{b0}$:

$$\boldsymbol{v}_0^I = \boldsymbol{v}_0^{\mathrm{A}} + \boldsymbol{\omega}_{b0} \times \boldsymbol{AI} = \boldsymbol{v}_{0\parallel}$$

Remember that $\boldsymbol{v}_{0\parallel}$ is parallel to $\boldsymbol{\omega}_{b0}$

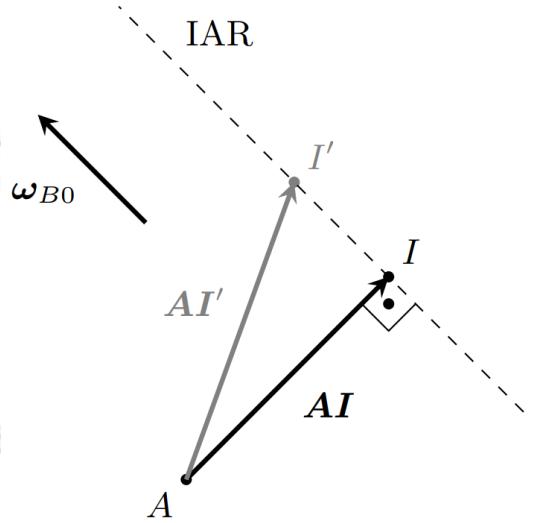
$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = (\boldsymbol{a} \cdot \boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a} \cdot \boldsymbol{b})\boldsymbol{c}$$

$$\boldsymbol{\omega}_{b0} \times \boldsymbol{v}_0^{\mathrm{A}} + (\boldsymbol{\omega}_{b0} \cdot \boldsymbol{AI})\boldsymbol{\omega}_{b0} - \omega_{b0}^2 \boldsymbol{AI} = \boldsymbol{0}$$

- This equation gives *all* points of the IAR. To find just one, let us further impose that $\boldsymbol{AI} \perp \boldsymbol{\omega}_{b0}$:

$$\boldsymbol{\omega}_{b0} \times \boldsymbol{v}_0^{\mathrm{A}} - \omega_{B0}^2 \boldsymbol{AI} = \boldsymbol{0}$$

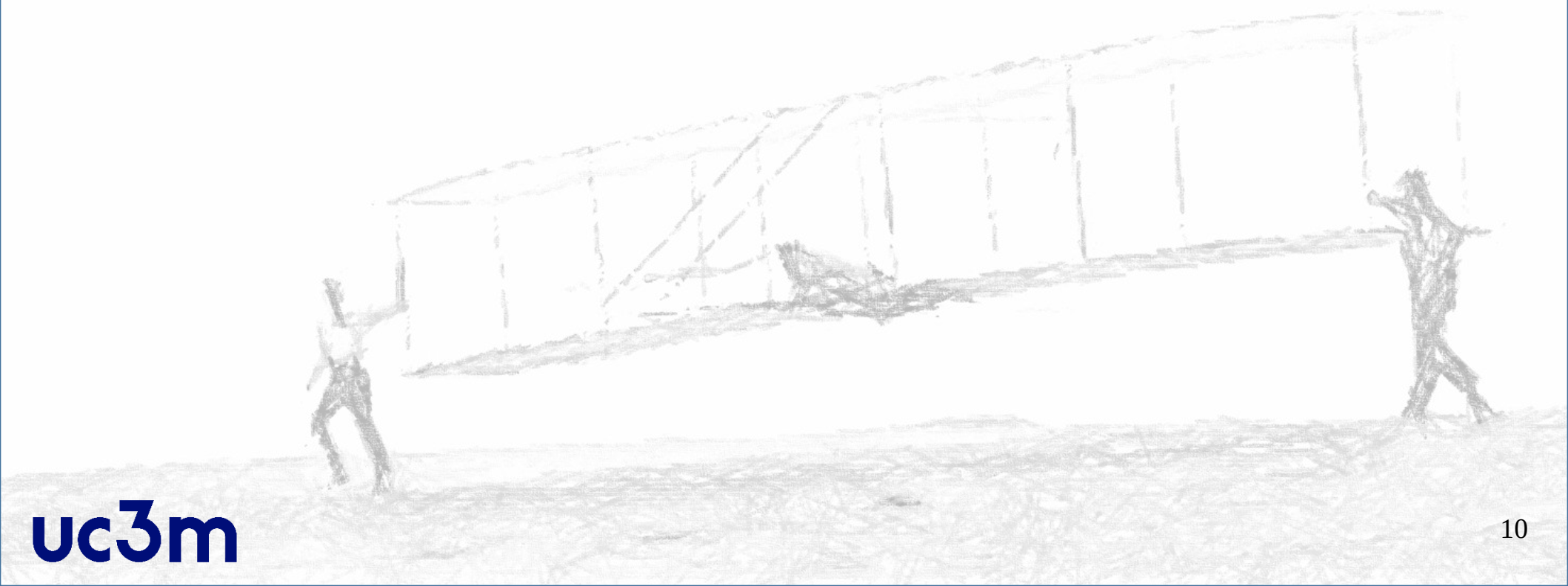
$$\boldsymbol{AI} = \frac{\boldsymbol{\omega}_{b0} \times \boldsymbol{v}_0^{\mathrm{A}}}{\omega_{b0}^2}$$



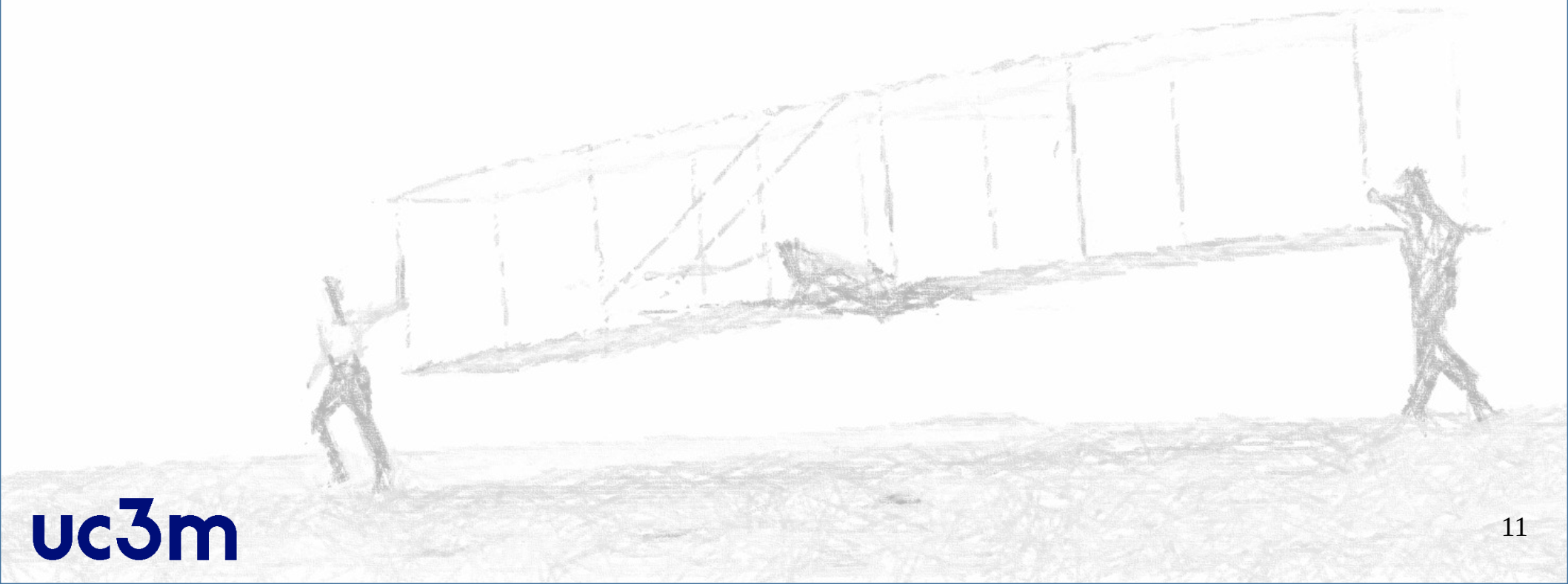
  - In the particular case that $\boldsymbol{v}_{0\parallel} = \boldsymbol{0}$, we have zero slip and we call it just "instantaneous axis of rotation"

Example

Three simultaneous measurements are made of the motion of a rigid airplane. The center of mass G, located at $\boldsymbol{r}_0^G = \mathbf{0}$ [m], has a velocity $\boldsymbol{v}_0^G = 4\boldsymbol{i} + 2\boldsymbol{j}$ [m/s]. Point A on the wing has $\boldsymbol{r}_0^A = -\boldsymbol{i} + 2\boldsymbol{j}$ [m] and $\boldsymbol{v}_0^A = 2\boldsymbol{i} + \boldsymbol{j} + 3\boldsymbol{k}$ [m/s]. Point C on the vertical tail has $\boldsymbol{r}_0^C = -4\boldsymbol{i} + \boldsymbol{k}$ [m] and $\boldsymbol{v}_0^C = 5\boldsymbol{i} - 3\boldsymbol{j} + 4\boldsymbol{k}$ [m/s].

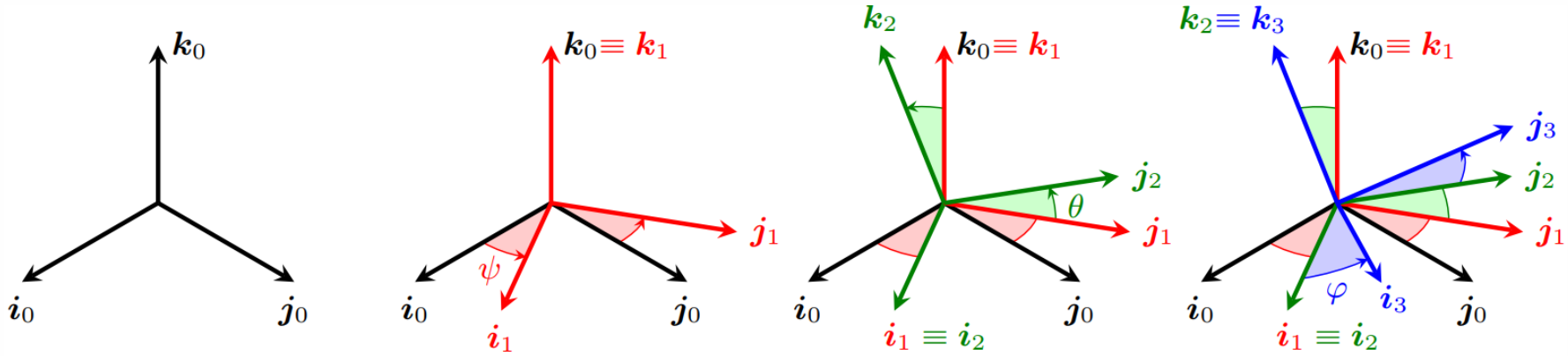


Example



Euler angles, revisited

- Thanks to the correspondence between rigid bodies and reference frames, we can describe the attitude of a rigid body with *Euler angles*



- Remember:
  - We can define Euler angles in different ways: 3-1-3, 3-2-1, etc
- Additional aspects of Euler angles:
  - Any attitude has multiple equivalent sets of Euler angles. For example, in the 3-1-3 Euler angles, the following values give the same attitude:

$$\psi = \frac{\pi}{2}; \theta = \frac{\pi}{2}; \varphi = 0 \qquad \psi = -\frac{\pi}{2}; \theta = \frac{\pi}{2}; \varphi = \pi$$

  - Certain attitudes are *singular,* with ψ, φ not defined (only the sum $\psi + \varphi$ is)
    - This happens whenever the second rotation aligns the first and third axes of rotation in inertial space. E.g. for 3-1-3 angles, this is $\theta = 0, \pi$

Summary

- A free rigid body is equivalent to a reference frame and has 6 degrees of freedom.
  - We need to specify its configuration (position + attitude) with 6 generalized coordinates
- A common choice is:
  - 3 coordinates to specify the position, velocity, and acceleration of a single body point A.
    - Next weeks, we will see that the center of mass G is a particularly good choice
  - 3 coordinates to specify the attitude, angular velocity, and angular acceleration
    - This can be done with Euler angles
  - Usually we define $S_b = \{A; B_b\}$ using this information
- Of course, this is not the only possibility: depending on the problem, other options might be more interesting
- Solving the kinematics of a rigid body amounts to providing general expressions for the position, velocity and acceleration of one point, the attitude of the rigid body, and its angular velocity and acceleration as a function of these 6 coordinates (and their derivatives):

$$\boldsymbol{r}_0^A, \boldsymbol{v}_0^A, \boldsymbol{a}_0^A \qquad \{\boldsymbol{i}_b, \boldsymbol{j}_b, \boldsymbol{k}_b\}$$
$$\boldsymbol{\omega}_{b0}, \boldsymbol{\alpha}_{b0}$$

Thank you!

