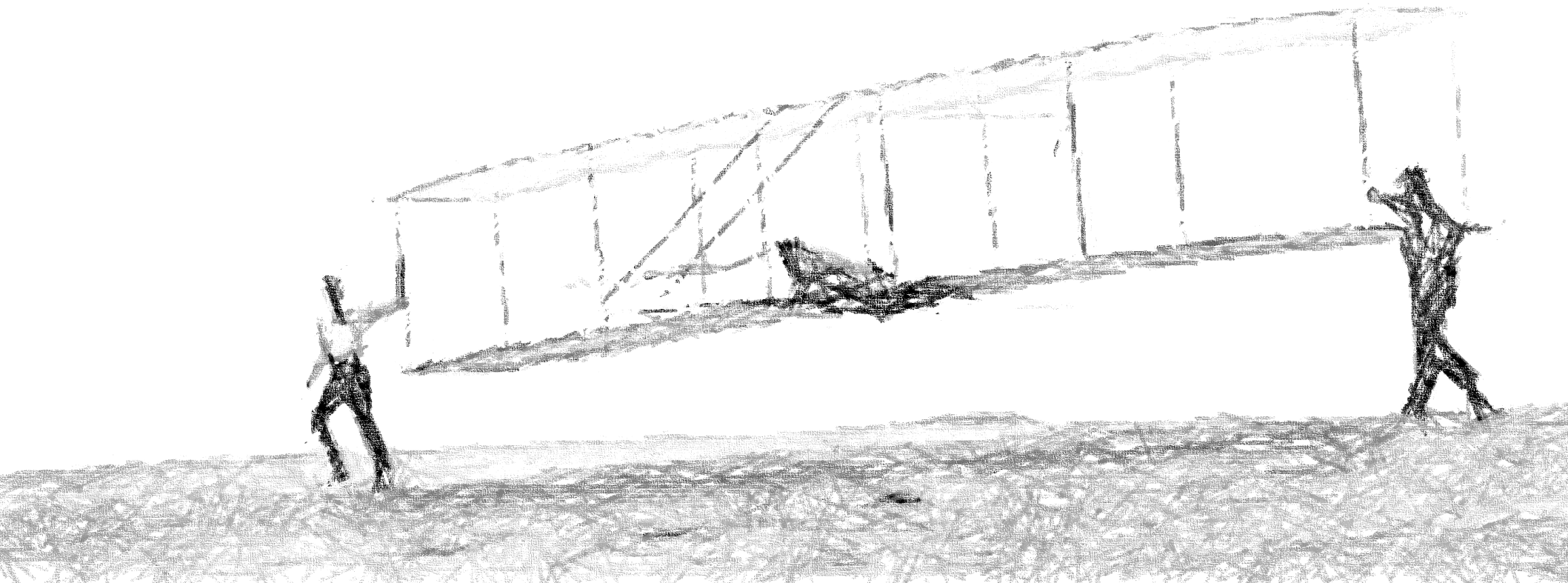
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Mechanics applied to Aerospace Engineering

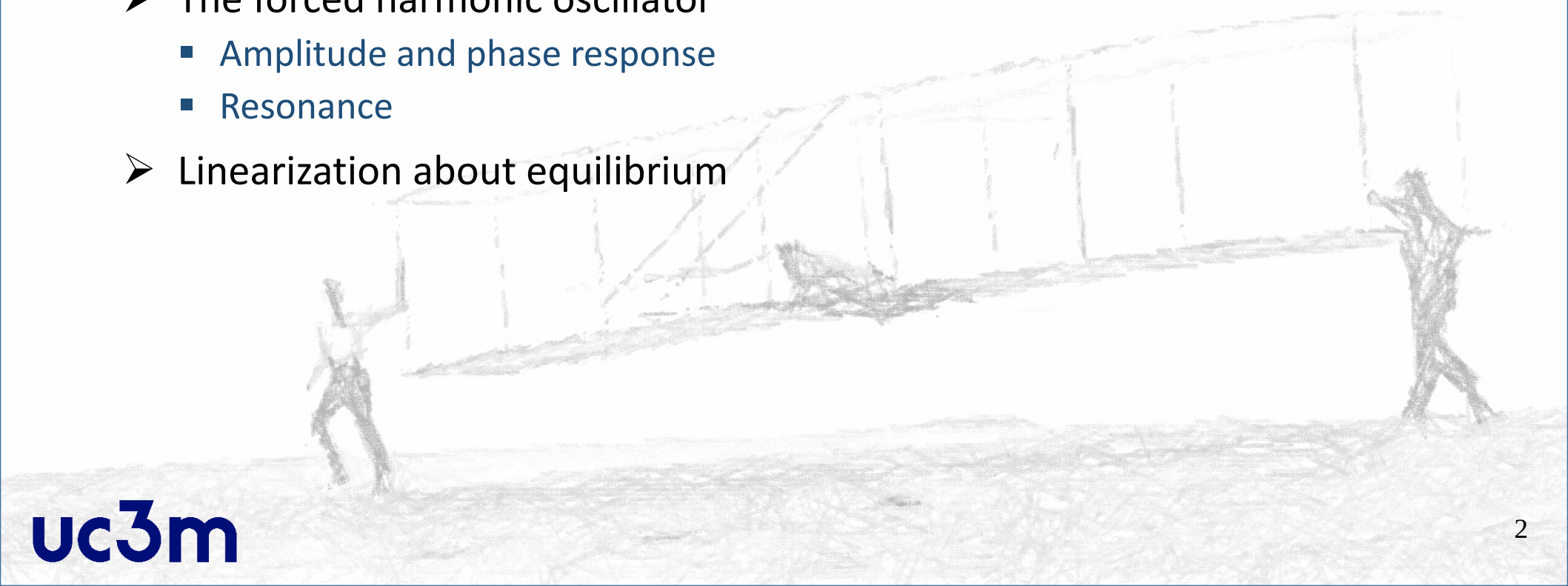
Course 251-14165 – BSc Aerospace Engineering

6. Oscillations



Contents

- Linear differential equations with constant coefficients
- Complex exponentials
- The free harmonic oscillator
  - Undamped case
  - Small damping
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- The forced harmonic oscillator
  - Amplitude and phase response
  - Resonance
- Linearization about equilibrium



Linear ODEs with constant coefficients

- Summary only: consider the second-order ODE with constant coefficients:

$$a\ddot{y} + b\dot{y} + cy = f(t)$$

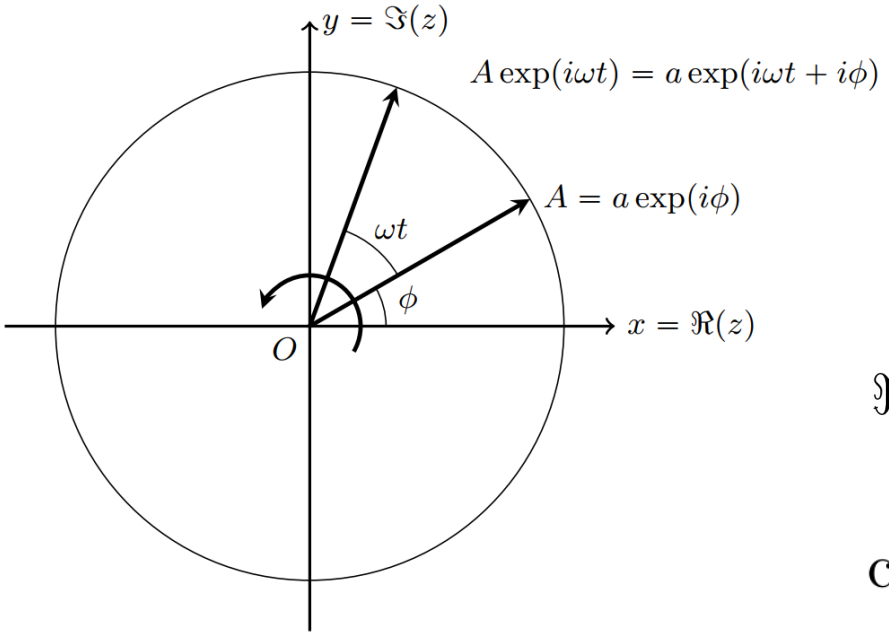
  - The equation $a\ddot{y} + b\dot{y} + cy = 0$ is known as the homogeneous equation
  - The equation with $f(t) \neq 0$ is the complete equation

- The solutions of the homogeneous problem form a two-dimensional vector space of functions
  - Characteristic equation: $ap^2 + bp + c = 0$. Complex roots: p_1, p_2
  - If $p_1 \neq p_2$, the homogeneous solution is $y_h(t) = A_1 \exp(p_1 t) + A_2 \exp(p_2 t)$
    - Functions $\{\exp(p_1 t), \exp(p_2 t)\}$ constitute a vector basis of the solution space
    - Constants A_1 and A_2 are the vector components of $y_h(t)$ in this basis
  - If $p_1 = p_2 (= p)$, the solution is $y_h(t) = A_1 \exp(pt) + A_2 t \exp(pt)$
- The solution space of the complete problem is a two-dimensional affine space
  - If $y_p(t)$ is a particular solution of the complete equation, then the general solution is $y(t) = y_h(t) + y_p(t)$
- Adding initial conditions $y(0) = y_0$ and $\dot{y}(0) = \dot{y}_0$ we have a Cauchy problem
  - A unique solution always exists (∃!)
  - Use the initial conditions to compute A_1 and A_2 in the general solution $y(t)$

Complex exponentials

- Summary only: consider $z = A\exp(pt)$ with $A, p \in \mathbb{C}$
  - We call $A = a\exp(i\phi)$ the complex amplitude
    - a is the (real, initial) amplitude and ϕ the initial phase angle
  - Let us call $p = -\gamma + i\omega$ the eigenvalue
    - ω is the angular frequency and γ the decay or damping rate

$$z = A\exp(pt) = a\exp(-\gamma t)\exp[i(\omega t + \phi)]$$



- Exponentials with $\omega > 0$ rotate counterclockwise
- Exponentials with $\gamma > 0$ decay with time
- Useful relations:

$$\Re(z) = \frac{z + z^*}{2} \qquad \Im(z) = \frac{z - z^*}{2i}$$

$$\cos b = \frac{e^{ib} + e^{-ib}}{2} \qquad \sin b = \frac{e^{ib} - e^{-ib}}{2i}$$

$$e^{ib} = \cos(b) + i\sin(b)$$

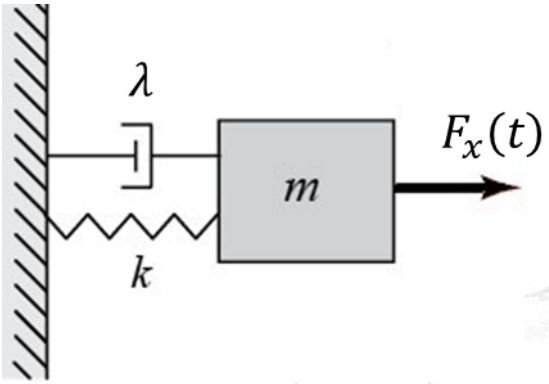
The harmonic oscillator

- Any physical system governed by the following equation of motion

$$a\ddot{y} + b\dot{y} + cy = f(t)$$

  is termed a *harmonic oscillator*

- The best-known example is a point particle P of mass m attached to the wall with a spring of stiffness constant k and zero natural length, plus a damper of damping constant λ. Additionally, there might be an external applied force $F_x(t)$:



  - The force of the spring is $-kx\boldsymbol{i}$
  - The force of the damper is $-\lambda\dot{x}\boldsymbol{i}$

$$m\ddot{x} + \lambda\dot{x} + kx = F_x(t)$$

  - We call $\omega_0 = \sqrt{k/m}$ the *natural angular frequency* of the (undamped) oscillator
  - We call $\gamma = \lambda/(2m)$ the *damping coefficient* or *damping ratio*

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = F_x(t)/m$$

- The characteristic equation $p^2 + 2\gamma p + \omega_0^2 = 0$ has the roots:

$$p = -\gamma \pm \sqrt{\gamma^2 - \omega_0^2}$$

Free oscillator; undamped (ideal) case

- We study first all cases with $F_x(t) = 0$. When $\lambda = 0$, there is no damping:

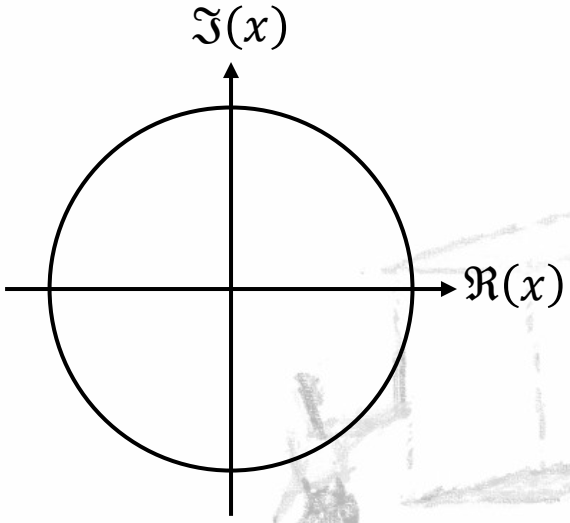
$$\ddot{x} + \omega_0^2 x = 0$$

- The solution of the characteristic equation is just $p = \pm i\omega_0$.
  Therefore, the general solution of the ODE is:

$$x(t) = A_1 \exp(i\omega_0 t) + A_2 \exp(-i\omega_0 t)$$

  - For a real solution, A_1 and A_2 must be complex conjugates:

$$A_1 = \frac{a}{2}\exp(i\phi)\,; \qquad A_2 = \frac{a}{2}\exp(-i\phi)$$



  - Alternative forms of writing the solution:

$$x(t) = a\frac{\exp(i\omega_0 t + i\phi) + \exp(-i\omega_0 t - i\phi)}{2}$$
$$= a\cos(\omega_0 t + \phi)$$
$$= C_1\cos(\omega_0 t) + C_2\sin(\omega_0 t)$$
$$\text{with: } C_1 = a\cos\phi\,;\, C_2 = -a\sin\phi$$

- Particularizing at $t = 0$ and using initial conditions x_0 and $\dot{x}_0$:

$$x_0 = C_1 = a\cos\phi\,; \qquad \dot{x}_0 = C_2\omega_0 = -a\omega_0\sin\phi$$

Undamped (ideal) case

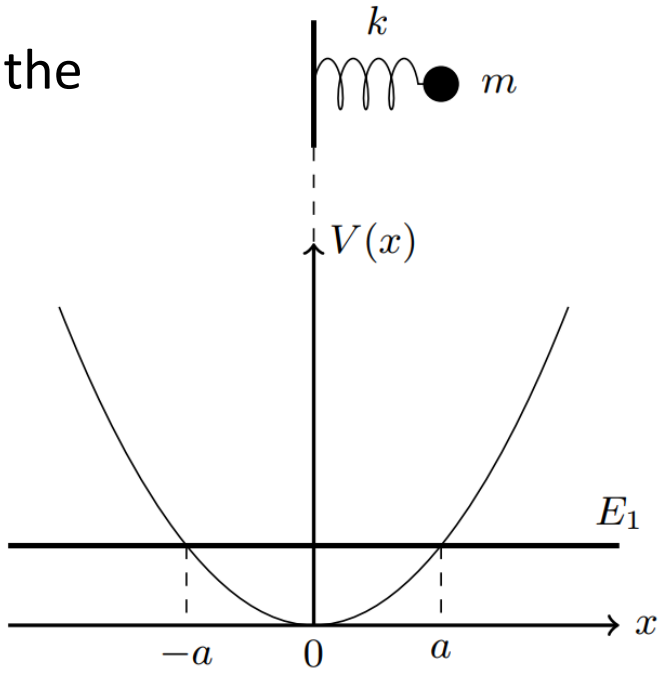
$$\ddot{x} + \omega_0^2 x = 0$$

- The problem is conservative with potential $V(x) = kx^2/2 = m\omega_0^2 x^2/2$

$$\frac{m\dot{x}^2}{2} + \frac{kx^2}{2} = E$$

- For a given E (computed from initial conditions), the turning points of the oscillator are:

$$x = \pm a = \pm\sqrt{2E/k}$$



- The period is:

$$\int_{-\sqrt{2E/k}}^{\sqrt{2E/k}} \frac{dx}{\sqrt{(2E - kx^2)/m}} = \frac{\tau}{2}$$

$$= \frac{1}{\omega_0} \arcsin\left(\frac{x}{\sqrt{2E/k}}\right)\Bigg]_{-\sqrt{2E/k}}^{\sqrt{2E/k}} = \frac{\pi}{\omega_0}$$

$$\Rightarrow \tau = \frac{2\pi}{\omega_0}$$

  - The (angular) frequency is ω_0; the (linear) frequency is $f = 1/\tau = \omega_0/(2\pi)$

Small damping

- Consider now that $\lambda \neq 0$, with $\gamma = \lambda/(2m) < \omega_0$

$$m\ddot{x} + \lambda\dot{x} + kx = 0$$

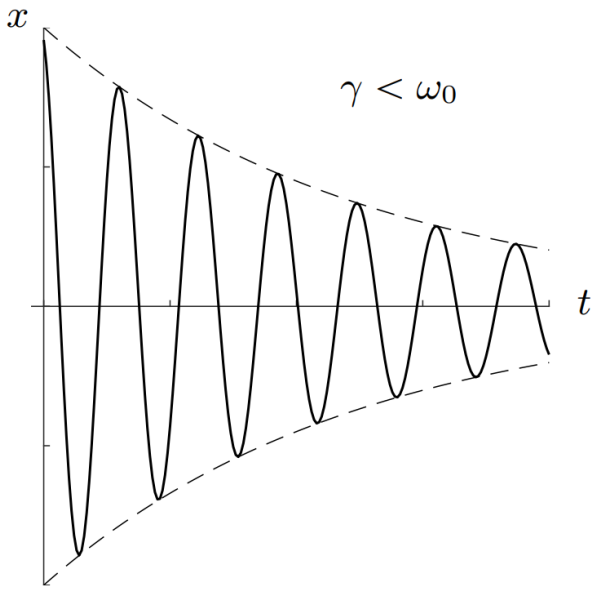
  - The eigenvalues are $p_1 = -\gamma + i\omega$; $p_2 = -\gamma - i\omega$, with:

$$\omega = \sqrt{\omega_0^2 - \gamma^2}$$

    where ω is the angular frequency of the *damped* harmonic oscillator

- The solution can be written in many alternative forms:

$$\begin{aligned} x &= \frac{1}{2}A\exp(i\omega t - \gamma t) + \frac{1}{2}A^*\exp(-i\omega t - \gamma t) \\ &= \Re\left[A\exp(i\omega t - \gamma t)\right] = a\exp(-\gamma t)\Re\left[\exp(i\phi)\exp(i\omega t)\right] \\ &= a\exp(-\gamma t)\cos(\omega t + \phi) \\ &= \exp(-\gamma t)[C_1\cos(\omega t) + C_2\sin(\omega t)]. \end{aligned}$$



  - We see the oscillatory behavior of the ideal case, with a slightly different angular frequency, and exponentially damped with decay rate γ
  - We define the relaxation time as $t_r = 1/\gamma$ and the quality factor $Q = \omega_0/(2\gamma)$

Large damping

- Consider now that $\gamma > \omega_0$

$$m\ddot{x} + \lambda\dot{x} + kx = 0$$

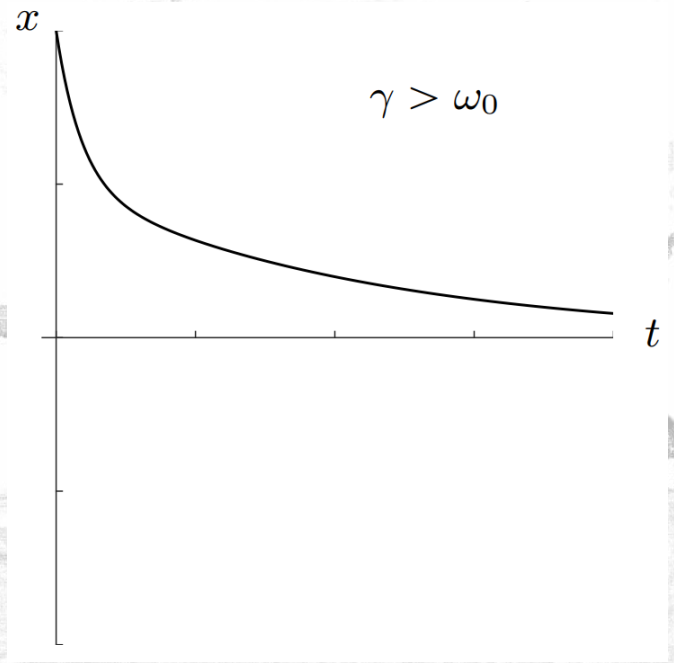
  - The eigenvalues are both real negative: $p_1 = -\gamma_+; p_2 = -\gamma_-$, with

$$\gamma_\pm = \gamma \pm \sqrt{\gamma^2 - \omega_0^2}$$

  - The oscillatory behavior has disappeared; we only have two real decaying exponentials:

$$x = C_+ \exp(-\gamma_+ t) + C_- \exp(-\gamma_- t)$$

    - A fast one, with γ_+, and a slow one with γ_- (which survives longer)
  - Like in other cases, the real constants C_+ and C_- must be computed from the initial conditions
  - The relaxation time is now $t_r \simeq 1/\gamma_-$



Critical damping

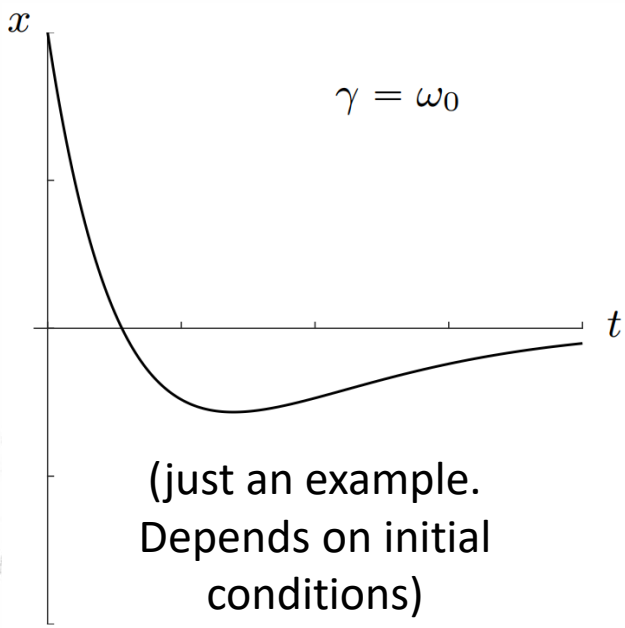
- When $\gamma = \omega_0$ then the two eigenvalues coincide: $p = -\gamma$

$$m\ddot{x} + \lambda\dot{x} + kx = 0$$

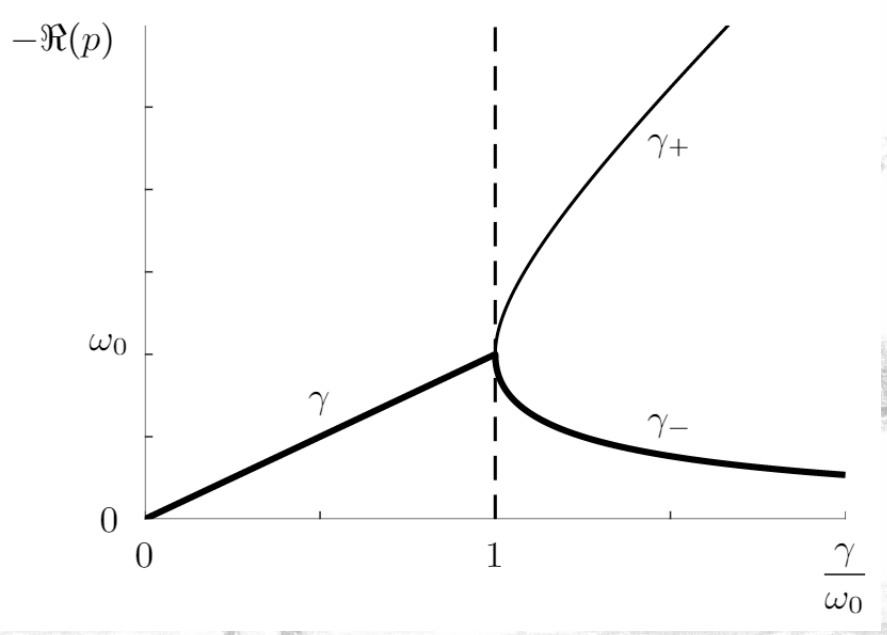
  - The general solution involves a new type of function, $t\exp(-\gamma t)$:

$$x = (a + bt)\exp(-\gamma t)$$

- Critical damping is optimal when we want the harmonic oscillator to settle back to its equilibrium configuration as quickly as possible



(just an example. Depends on initial conditions)



The smaller decay rate (of the two independent solutions) is maximal in the critical damping case

The forced harmonic oscillator

- We now consider cases where $F_x(t) \neq 0$. We will focus on cases with:

$$m\ddot{x} + \lambda\dot{x} + kx = F_1 \cos(\omega_1 t)$$

where F_1, ω_1 are given constants.

- To simplify our calculations, we shall work with the following complex equation and then just take the real part ($x = \Re(z)$):

$$m\ddot{z} + \lambda\dot{z} + kz = F_1 \exp(i\omega_1 t)$$

  - To find the general solution, we must find one particular solution (and add it to the homogeneous solution)

$$z(t) = z_h(t) + z_p(t)$$

Let us try functions of the form: $z_p(t) = A_1 \exp(i\omega_1 t)$,
with $A_1 = a_1 \exp(i\phi_1)$. Entering with $z_p(t)$ in the differential equation:

$$\left(-m\omega_1^2 + i\lambda\omega_1 + k\right) A_1 = F_1$$

$$\left(\omega_0^2 - \omega_1^2 + 2i\gamma\omega_1\right) a_1 = \frac{F_1}{m} \exp(-i\phi_1)$$

$$a_1 = \frac{F_1/m}{\sqrt{(\omega_0^2 - \omega_1^2)^2 + 4\gamma^2\omega_1^2}}$$

$$\left(\omega_0^2 - \omega_1^2\right) a_1 = \frac{F_1}{m} \cos\phi_1$$

$$\tan\phi_1 = -\frac{2\gamma\omega_1}{\omega_0^2 - \omega_1^2}$$

$$2\gamma\omega_1 a_1 = -\frac{F_1}{m} \sin\phi_1$$

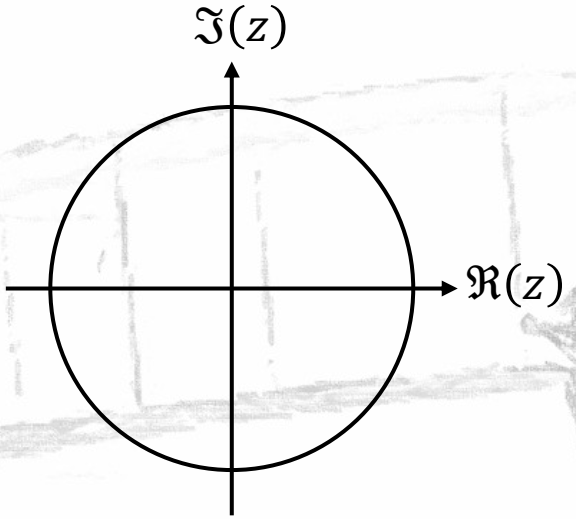
The forced harmonic oscillator

$$a_1 = \frac{F_1/m}{\sqrt{(\omega_0^2 - \omega_1^2)^2 + 4\gamma^2\omega_1^2}} \qquad \tan\phi_1 = -\frac{2\gamma\omega_1}{\omega_0^2 - \omega_1^2}$$

- General solution of the complex equation:

$$z(t) = z_h(t) + a_1 \exp(i\omega_1 t + i\phi_1)$$

  - Our particular solution has the same angular frequency as the force
  - The angle ϕ_1 is the phase difference between the oscillator and the force. It is always ≤ 0, so the oscillator *lags* behind the force
  - As long as $\lambda \neq 0$, $x_h(t)$ dies with time and only $x_p(t)$ survives: the particular solution corresponds to the solution at long times and is independent of the initial conditions



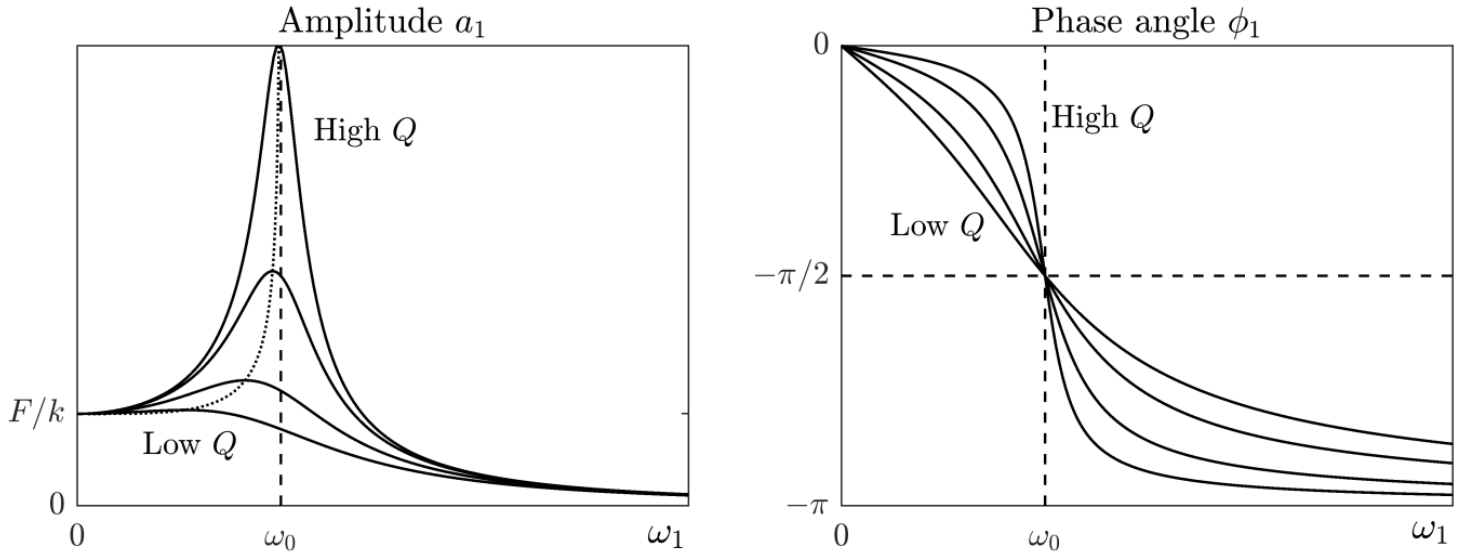
- Taking the real part:

$$x(t) = a\exp(-\gamma t)\cos(\omega t + \phi) + a_1\cos(\omega_1 t + \phi_1)$$

The forced harmonic oscillator

$$a_1 = \frac{F_1/m}{\sqrt{(\omega_0^2 - \omega_1^2)^2 + 4\gamma^2\omega_1^2}} \qquad \tan\phi_1 = -\frac{2\gamma\omega_1}{\omega_0^2 - \omega_1^2}$$

- The amplitude a_1 and the phase difference ϕ_1 vary wildly with $\omega_0^2 - \omega_1^2$:



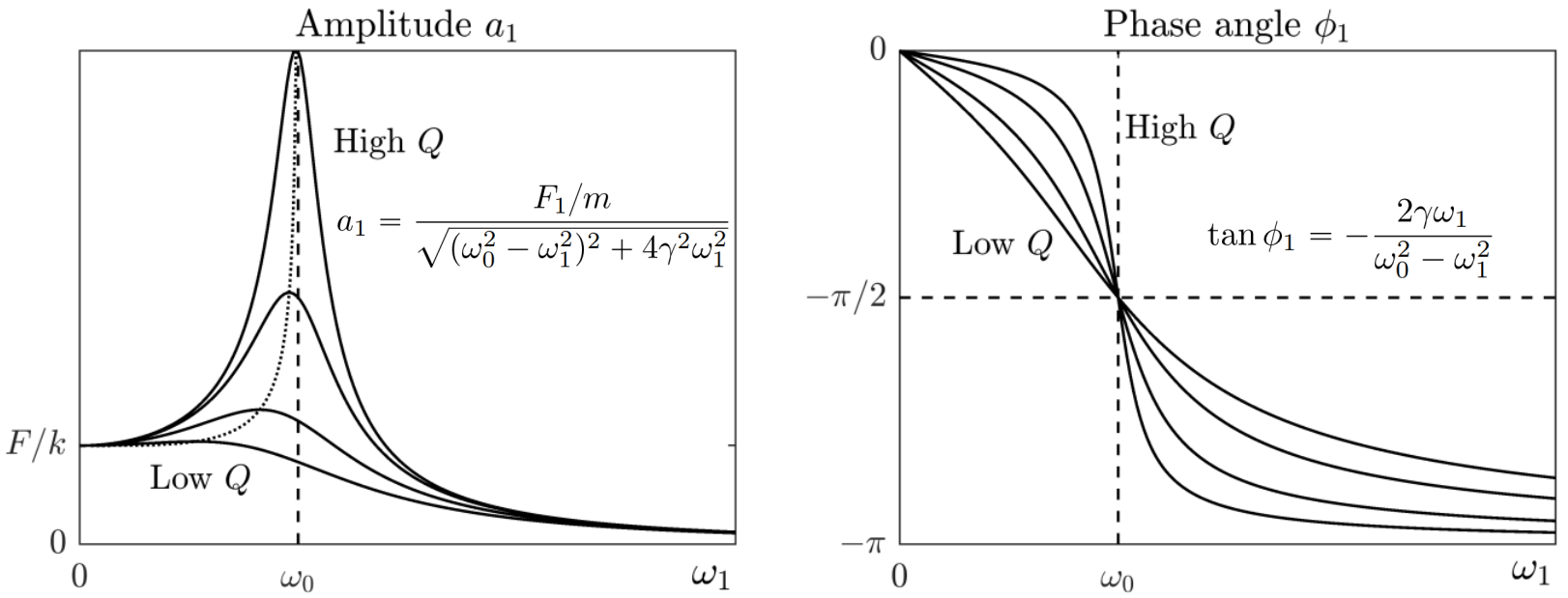
- When $\omega_1 \ll \omega_0$, x is approximately the instantaneous equilibrium position, as if F_x were constant, and the phase difference is small:

$$m\ddot{x} + \lambda\dot{x} + kx = F_1\cos(\omega_1 t) \quad \Rightarrow \quad a_1 \simeq F_1/k\,, \phi_1 \simeq 0$$

- When $\omega_1 \gg \omega_0$, the dominant term in the left-hand side is $m\ddot{x}$, and the oscillator is completely out of phase with the force:

$$m\ddot{x} + \lambda\dot{x} + kx = F_1\cos(\omega_1 t) \quad \Rightarrow \quad a_1 \simeq F_1/(m\omega_1^2)\,, \phi_1 \simeq -\pi$$

Resonance



- If we fix ω_0 and γ, then a_1 has a maximum with ω_1 at:

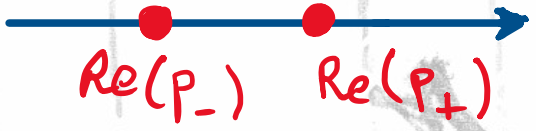
$$\omega_1 = \sqrt{\omega_0^2 - 2\gamma^2}$$

  - This is the *resonant frequency*, and for $\gamma \ll \omega_0$ it is $\omega_1 \simeq \omega_0$. We usually want to avoid designing systems that operate near their resonance freq.
  - The approximate half-width of the peak is γ
  - The peak does not exist when $\gamma \geq \omega_0/\sqrt{2}$
  - The phase difference at $\omega_1 = \omega_0$ is always $\phi_1 = -\pi/2$

Linearization about equilibrium

- Consider a 1-dimensional system with a nonlinear equation of motion
$$m\ddot{x} = F(x, \dot{x})$$
with an equilibrium configuration at $x = x^*$. This means $F(x^*, 0) = 0$
- We take $\xi = x - x^*$ ($\dot{x} = \dot{\xi}$, $\ddot{x} = \ddot{\xi}$) and expand F about $(x^*, 0)$:
$$F(x^* + \xi, \dot{\xi}) = \cancel{F(x^*, 0)} + \left(\frac{\partial F}{\partial x}\bigg|_{(x^*,0)}\right)\xi + \left(\frac{\partial F}{\partial \dot{x}}\bigg|_{(x^*,0)}\right)\dot{\xi} + \cdots$$

- Keeping only linear terms we can write for ξ sufficiently small:
$$m\ddot{\xi} = -k\xi - \lambda\dot{\xi} \qquad \Rightarrow \qquad m\ddot{\xi} + \lambda\dot{\xi} + k\xi = 0$$
  - This is a 2-order ODE with constant coefficients! Just like a harmonic oscillator. The eigenvalues are:
$$p_{\pm} = -\frac{\lambda}{2m} \pm \sqrt{\frac{\lambda^2}{4m^2} - \frac{k}{m}}$$



  - If $\Re(p_+) < 0$, any perturbation from equilibrium dies away over time. The equilibrium is *asymptotically stable*
  - If $\Re(p_+) = 0$ and $p_+ \neq p_-$, the perturbation survives over time but does not grow unbounded. The equilibrium is *marginally stable* (but, we must check higher-order terms to be sure)
  - Else, the perturbation grows over time. Equilibrium is *unstable*

Thank you!

