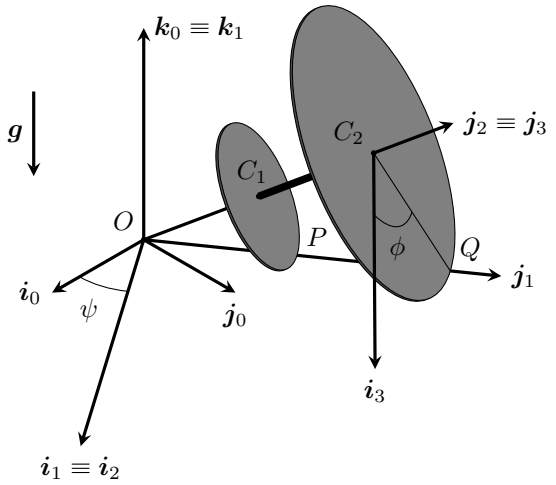
Two thin, heavy, homogeneous wheels, of center C_1 and C_2 and radius a and $2a$ respectively, are connected to each other with a massless, perpendicular axle of length $\sqrt{3}a$ forming a single rigid body b as shown in the figure. The surface density of the wheel material is σ. We label the points of the rigid body that are in contact with the ground at each instant as P and Q. The friction coefficient μ between the wheels and the ground is sufficiently large to ensure that the wheels always roll without slipping. We define the inertial reference frame S_0 and the auxiliary, rotating reference frame S_1 as shown. We also define other auxiliary vectors in the figure, $\boldsymbol{i}_2, \boldsymbol{j}_2, \boldsymbol{i}_3, \boldsymbol{j}_3$. The axis Oy_1 follows points P and Q as they move. The angle ψ is measured between the axes Ox_0 and Ox_1. The angle ϕ is the angle that the wheels have rotated about the axle since the initial instant of time. Initially, $\psi = 0$, $\phi = 0$, and vector $\boldsymbol{i}_3$ points in the direction of $\boldsymbol{C_2Q}$.
Hint: Observe that the points O, C_1, and C_2 are always aligned. Remember to define all the vector bases that you use as a function of the vectors $\boldsymbol{i}_0, \boldsymbol{j}_0, \boldsymbol{k}_0$, or other, already defined vectors.



Before we start solving the problem, we note that the rigid body b has only one effective degree of freedom. Also, we define the following auxiliary vector bases. If any of the following vectors (or others) are used in the solution, they must first be defined. B_1 is given by:

$$\boldsymbol{i}_1 = \cos(\psi)\boldsymbol{i}_0 + \sin(\psi)\boldsymbol{j}_0; \qquad \boldsymbol{j}_1 = -\sin(\psi)\boldsymbol{i}_0 + \cos(\psi)\boldsymbol{j}_0; \qquad \boldsymbol{k}_1 = \boldsymbol{k}_0.$$

B_2 is given by:

$$\boldsymbol{i}_2 = \boldsymbol{i}_1; \qquad \boldsymbol{j}_2 = \frac{\sqrt{3}}{2}\boldsymbol{j}_1 + \frac{1}{2}\boldsymbol{k}_1; \qquad \boldsymbol{k}_2 = -\frac{1}{2}\boldsymbol{j}_1 + \frac{\sqrt{3}}{2}\boldsymbol{k}_1.$$

B_3 is given by:

$$\boldsymbol{i}_3 = \sin(\phi)\boldsymbol{i}_2 - \cos(\phi)\boldsymbol{k}_2; \qquad \boldsymbol{j}_3 = \boldsymbol{j}_2 \qquad \boldsymbol{k}_3 = \cos(\phi)\boldsymbol{i}_2 + \sin(\phi)\boldsymbol{k}_2.$$

Basis B_3 is body-fixed, i.e., it rotates with the rigid body, b.

1. Compute the total mass m, the distance between point C_1 on the axle and the center of mass G, and the tensor of inertia about G of the object. (**1.5 points**).

   Hint: the tensor of inertia of a thin, homogeneous disk of mass M and radius L about its center C, expressed in a vector basis **with the second unit vector perpendicular to the disk**, is given by:

$$\overline{\overline{I}}_C = \frac{ML^2}{4}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The total mass is $m = \sigma\pi a^2 + 4\sigma\pi a^2 = 5\sigma\pi a^2$. The distance between C_1 and the center of mass G is $d = 4\sqrt{3}a/5$. The tensor of inertia of the body b can be computed using Steiners theorem and the tensor of inertia of each disk about its center. In vector basis B_2:

$$\begin{aligned}\overline{\overline{I}}^{\,b}_G &= \frac{\sigma\pi a^4}{4}\begin{bmatrix}1&0&0\\0&2&0\\0&0&1\end{bmatrix}_2 + \frac{48\sigma\pi a^4}{25}\begin{bmatrix}1&0&0\\0&0&0\\0&0&1\end{bmatrix}_2 + 4\sigma\pi a^4\begin{bmatrix}1&0&0\\0&2&0\\0&0&1\end{bmatrix}_2 + \frac{12\sigma\pi a^4}{25}\begin{bmatrix}1&0&0\\0&0&0\\0&0&1\end{bmatrix}_2\\ &= \frac{\sigma\pi a^4}{20}\begin{bmatrix}133&0&0\\0&170&0\\0&0&133\end{bmatrix}_2 = \begin{bmatrix}A&0&0\\0&B&0\\0&0&A\end{bmatrix}_2\end{aligned}$$

In the rest of the problem, you can name the resulting inertia tensor components with letters A, B, etc. and use them instead of their numerical expressions.

2. Write down the position, velocity, and acceleration of G, and the angular velocity and acceleration of the rigid body b, with respect to S_0 (**2 points**).

   The relevant angular velocities of the problem are:

   $$\boldsymbol{\omega}_{10} = \dot{\psi}\boldsymbol{k}_1; \qquad \boldsymbol{\omega}_{21} = \boldsymbol{0}; \qquad \boldsymbol{\omega}_{32} = -\dot{\phi}\boldsymbol{j}_2; \qquad \boldsymbol{\omega}_{30} = (\dot{\psi}/2 - \dot{\phi})\boldsymbol{j}_2 + \sqrt{3}\dot{\psi}\boldsymbol{k}_2/2;$$

   The angular acceleration is:

   $$\boldsymbol{\alpha}_{30} = \sqrt{3}\dot{\psi}\dot{\phi}\boldsymbol{i}_2/2 + (\ddot{\psi}/2 - \ddot{\phi})\boldsymbol{j}_2 + \sqrt{3}\ddot{\psi}\boldsymbol{k}_2/2;$$

   The distance from O to C_1 is easily found from the right triangle OC_1P. The center of mass has:

   $$\begin{aligned}\boldsymbol{r}_0^G &= 27a\boldsymbol{j}_1/10 + 9\sqrt{3}a\boldsymbol{k}_1/10\\ \boldsymbol{v}_0^G &= -27a\dot{\psi}\boldsymbol{i}_1/10\\ \boldsymbol{a}_0^G &= -27a\ddot{\psi}\boldsymbol{i}_1/10 - 27a\dot{\psi}^2\boldsymbol{j}_1/10\end{aligned}$$

3. Find out the kinematic relation between ψ and ϕ (**1.5 points**).

   The rolling without slip condition at P and Q sets a very strong constraint on the motion of the rigid body. We need to work with the following two equations, derived from the velocity field of the rigid body:

   $$\begin{aligned}\boldsymbol{v}_0^P &= \boldsymbol{v}_0^G + \boldsymbol{\omega}_{31}\times\boldsymbol{GP} = \boldsymbol{0}\\ \boldsymbol{v}_0^Q &= \boldsymbol{v}_0^G + \boldsymbol{\omega}_{31}\times\boldsymbol{GQ} = \boldsymbol{0}\end{aligned}$$

   From these expressions one finds $2\dot{\psi} = \dot{\phi}$. This constraint is integrable into $2\psi = \phi$. The configuration of the body can be described in terms of a single variable.

4. Identify and write down the general expression of all the forces acting on the rigid body, and reduce them to the resultant force and resultant moment of force about G. (**1.5 points**).

   The only applied force is the weight $\boldsymbol{W} = -mg\boldsymbol{k}_1$. There are 3 unknown reaction forces at P and 3 unknown reaction forces at Q:

   $$\boldsymbol{F}_P = F_{xP}\boldsymbol{i}_1 + F_{yP}\boldsymbol{j}_1 + N_P\boldsymbol{k}_1; \qquad \boldsymbol{F}_Q = F_{xQ}\boldsymbol{i}_1 + F_{yQ}\boldsymbol{j}_1 + N_Q\boldsymbol{k}_1.$$

   These forces can be reduced to the resultant force $\boldsymbol{\mathcal{F}}$ (the sum of them) and the resultant moment at G, $\boldsymbol{\mathcal{M}}_G$,

   $$\begin{aligned}\boldsymbol{\mathcal{M}}_G = {}& [13N_Q - 7N_P + 9\sqrt{3}(F_{yP} + F_{yQ})]a\boldsymbol{i}_2/10\\ & - [F_{xP} + 2F_{xQ}]a\boldsymbol{j}_2\\ & + [4F_{xP} - F_{xQ}]\sqrt{3}a\boldsymbol{k}_2/5\end{aligned}$$

5. Write down the equations of motion for the rigid body in the form of 6 scalar differential equations (**1.5 points**).

   We project the traslational equation of motion of the center of mass along the vectors of basis B_1:

$$\begin{aligned} -27am\ddot{\psi}/10 &= F_{xP} + F_{xQ}; \\ -27am\dot{\psi}^2/10 &= F_{yP} + F_{yQ}; \\ 0 &= -mg + N_P + N_Q. \end{aligned}$$

   The angular momentum vector of b about G with respect to S_0 is:

$$\boldsymbol{H}^b_{G0} = B(\dot{\psi}/2 - \dot{\phi})\boldsymbol{j}_2 + \sqrt{3}A\dot{\psi}\boldsymbol{k}_2/2 = -3B\dot{\psi}\boldsymbol{j}_2/2 + \sqrt{3}A\dot{\psi}\boldsymbol{k}_2/2,$$

   and the rotational equation of motion of the rigid body about G,

$$\left.\frac{\mathrm{d}\boldsymbol{H}^b_{G0}}{\mathrm{d}t}\right|_2 + \boldsymbol{\omega}_{20} \times \boldsymbol{H}^b_{G0} = \boldsymbol{\mathcal{M}}_G,$$

   projected along the vectors of basis B_2 gives:

$$\begin{aligned} (A+3B)\sqrt{3}\dot{\psi}^2/4 &= [13N_Q - 7N_P + 9\sqrt{3}(F_{yP} + F_{yQ})]a/10 \\ -3B\ddot{\psi}/2 &= -[F_{xP} + 2F_{xQ}]a \\ A\sqrt{3}\ddot{\psi}/2 &= [4F_{xP} - F_{xQ}]\sqrt{3}a/5 \end{aligned}$$

   From these equations is easy to see that $\ddot{\psi} = 0$ and $F_{xP} = F_{xQ} = 0$.

6. Find out the maximum value of $\dot{\psi}$ for which both wheels remain in contact with the ground (**1 point**).

   The smaller wheel will leave the ground when $N_P = 0$. Imposing this condition on the first rotational equation, using the rest of equations to reduce the problem to an equation on N_P and $\dot{\psi}$,

$$\frac{10(A+3B)}{ma^2}\sqrt{3}\dot{\psi}^2/4 + 9\sqrt{3}(27\dot{\psi}^2/10) = 13\frac{g}{a} \qquad \Rightarrow \dot{\psi}^2_{\max} = \frac{104g}{323\sqrt{3}a}$$

7. Discuss whether the mechanical energy of the rigid body in S_0 is conserved in this problem. Note that you do not need to write down the equation of mechanical energy (**0.5 points**).

   Yes, the mechanical energy of b is conserved in S_0, because none of the forces exert work. In fact, since the potential energy is constant (weight exerts no work), the kinetic energy is conserved.

8. Is the system of equations of question 5 closed? Why, or why not? (**0.5 point**).

   The way we have defined the reaction forces, the system has 7 unknowns and 6 equations and therefore is not closed. The problem is due to the F_{yP} and F_{yQ} reactions: they act along the same line. Thus, we can only solve for the sum $F_{yP} + F_{yQ}$, but not each individual force. Observe that only this sum appears in the equations.