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nsabenocontesta

Mecánica aplicada a la Ingeniería Aeroespacial

2º Grado en Ingeniería Aeroespacial

Escuela Politécnica Superior. Campus de Leganés
Universidad Carlos III de Madrid

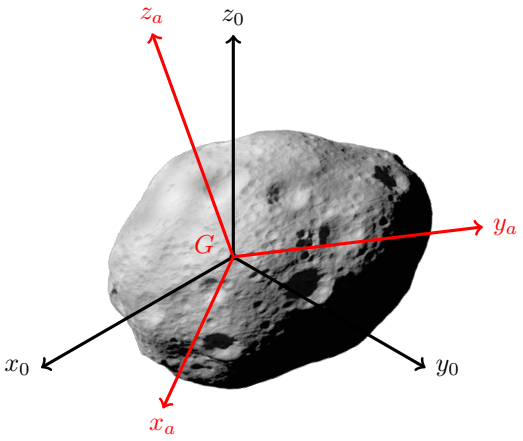
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An irregularly-shaped, rigid asteroid a with center of mass G is floating freely in space. We define an inertial reference frame $S_0 : \{G; B_0\}$, and a body-fixed reference frame $S_a : \{G; B_a\}$ as shown in the figure, where $B_0 = \{\boldsymbol{i}_0, \boldsymbol{j}_0, \boldsymbol{k}_0\}$ and $B_a = \{\boldsymbol{i}_a, \boldsymbol{j}_a, \boldsymbol{k}_a\}$. One can reach B_a from B_0 by a sequence of Euler rotations 1-2-3, with angles that we shall call ψ, θ, and ϕ (in that order). The asteroid has mass m and a tensor of inertia at G that expressed in the vector basis B_a is

$$\overline{\overline{\mathcal{I}}}_G = m\ell^2 \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 8 \end{bmatrix}_a ,$$

where ℓ is a characteristic length of the asteroid. Initially, $(\psi_0, \theta_0, \phi_0) = (\pi/3, 0, \pi/2)$ and $(\dot{\psi}_0, \dot{\theta}_0, \dot{\phi}_0) = (-2, 0, 1)/\tau$, where τ is a characteristic time.



1. Express the vectors $\boldsymbol{i}_a$, $\boldsymbol{j}_a$, $\boldsymbol{k}_a$ in the vector basis B_0 as a function of ψ, θ, and ϕ **(1.5 points)**.

   We use two auxiliary reference frames as scaffolding to reach B_a from B_0

   $$\begin{cases} \boldsymbol{i}_1 = \boldsymbol{i}_0 \\ \boldsymbol{j}_1 = \cos\psi \boldsymbol{j}_0 + \sin\psi \boldsymbol{k}_0 \\ \boldsymbol{k}_1 = -\sin\psi \boldsymbol{j}_0 + \cos\psi \boldsymbol{k}_0 \end{cases} \quad \begin{cases} \boldsymbol{i}_2 = \cos\theta \boldsymbol{i}_1 - \sin\theta \boldsymbol{k}_1 \\ \boldsymbol{j}_2 = \boldsymbol{j}_1 \\ \boldsymbol{k}_2 = \sin\theta \boldsymbol{i}_1 + \cos\theta \boldsymbol{k}_1 \end{cases} \quad \begin{cases} \boldsymbol{i}_a = \cos\phi \boldsymbol{i}_2 + \sin\phi \boldsymbol{j}_2 \\ \boldsymbol{j}_a = -\sin\phi \boldsymbol{i}_2 + \cos\phi \boldsymbol{j}_2 \\ \boldsymbol{k}_a = \boldsymbol{k}_2 \end{cases} \tag{1}$$

   This is enough to answer the question. With a little extra algebra one obtains:

   $$\begin{cases} \boldsymbol{i}_a = \cos\phi\cos\theta \boldsymbol{i}_0 + (\sin\phi\cos\psi + \cos\phi\sin\theta\sin\psi)\boldsymbol{j}_0 + (\sin\phi\sin\psi - \cos\phi\sin\theta\cos\psi)\boldsymbol{k}_0 \\ \boldsymbol{j}_a = -\sin\phi\cos\theta \boldsymbol{i}_0 + (\cos\phi\cos\psi - \sin\phi\sin\theta\sin\psi)\boldsymbol{j}_0 + (\cos\phi\sin\psi + \sin\phi\sin\theta\cos\psi)\boldsymbol{k}_0 \\ \boldsymbol{k}_a = \sin\theta \boldsymbol{i}_0 - \cos\theta\sin\psi \boldsymbol{j}_0 + \cos\theta\cos\psi \boldsymbol{k}_0 \end{cases} \tag{2}$$

2. Write down the angular velocity vector of the asteroid, $\boldsymbol{\omega}_{a0}$, in basis B_a as a function of ψ, θ, ϕ and their derivatives **(1 point)**.

   The angular velocity vector $\boldsymbol{\omega}_{a0}$ equals the sum of the angular velocity of each simple rotation:

   $$\boldsymbol{\omega}_{a0} = \dot{\psi}\boldsymbol{i}_1 + \dot{\theta}\boldsymbol{j}_2 + \dot{\phi}\boldsymbol{k}_a = (\dot{\psi}\cos\theta\cos\phi + \dot{\theta}\sin\phi)\boldsymbol{i}_a - (\dot{\psi}\cos\theta\sin\phi - \dot{\theta}\cos\phi)\boldsymbol{j}_a + (\dot{\psi}\sin\theta + \dot{\phi})\boldsymbol{k}_a \tag{3}$$

In the following, we shall denote the components of $\boldsymbol{\omega}_{a0}$ in B_a as ω_{xa}, ω_{ya}, ω_{za}, and define a point P with position vector $\boldsymbol{r}_a^P = x_a\boldsymbol{i}_a + y_a\boldsymbol{j}_a + z_a\boldsymbol{k}_a = \tau\ell(\omega_{xa}\boldsymbol{i}_a + \omega_{ya}\boldsymbol{j}_a + \omega_{za}\boldsymbol{k}_a)$.

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3. Compute the kinetic energy T_0 of the asteroid. Demonstrate that P must remain on the surface of a body-fixed ellipsoid at all times. What are the sizes of the semi-axes of this ellipsoid? **(2.5 points)**.

   Since there are no forces or torques exerting work on the body, the kinetic energy is conserved:

$$T_0 = \frac{1}{2}\boldsymbol{\omega}_{a0} \cdot \bar{\bar{I}}_G \cdot \boldsymbol{\omega}_{a0} = m\ell^2(\omega_{xa}^2 + 2\omega_{ay}^2 + 4\omega_{az}^2) = 12\frac{m\ell^2}{\tau^2} \tag{4}$$

$$\Rightarrow \frac{x_a^2}{12\ell^2} + \frac{y_a^2}{6\ell^2} + \frac{z_a^2}{3\ell^2} = 1 \tag{5}$$

   which is the implicit equation of an ellipsoid in the Cartesian coordinates of P in S_a, with semi-axes $\sqrt{12}\ell$, $\sqrt{6}\ell$, and $\sqrt{3}\ell$.

4. Compute the angular momentum vector $\boldsymbol{H}_{G,0}$ of the asteroid. Demonstrate that the component of $\boldsymbol{\omega}_{a0}$ in the direction of $\boldsymbol{H}_{G,0}$ is constant. Show that P must remain on an invariant plane of S_0 at all times, and compute the distance to this plane from G. **(2 points)**.

   Since there are no external torques, $\boldsymbol{H}_{G,0}$ is a constant vector in S_0 (inertial space)

$$\boldsymbol{H}_{G,0} = \bar{\bar{I}}_G \cdot \boldsymbol{\omega}_{a0} = m\ell^2(2\omega_{xa}\boldsymbol{i}_a + 4\omega_{ya}\boldsymbol{j}_a + 8\omega_{za}\boldsymbol{k}_a) = \frac{8m\ell^2}{\tau}(\boldsymbol{j}_a + \boldsymbol{k}_a). \tag{6}$$

   It is easy to see that

$$T_0 = \frac{1}{2}\boldsymbol{\omega}_{a0} \cdot \boldsymbol{H}_{G,0}. \tag{7}$$

   As such, the projection of $\boldsymbol{\omega}_{a0}$ (and $\boldsymbol{r}_a^P$) along the inertial direction defined by $\boldsymbol{H}_{G,0}$ is constant. This means that P must remain in a plane perpendicular to $\boldsymbol{H}_{G,0}$, at a distance $\boldsymbol{r}_a^P \cdot \boldsymbol{H}_{G,0}/H_{G,0} = 3\ell/\sqrt{2}$ from G. Calling $\boldsymbol{r}_a^P = x_0\boldsymbol{i}_0 + y_0\boldsymbol{j}_0 + z_0\boldsymbol{k}_0$, the equation for the plane is

$$-x_0 - \frac{\sqrt{3}}{2}y_0 + \frac{1}{2}z_0 = 3\ell \tag{8}$$

Since the ellipsoid is body-fixed, it rotates with the same angular velocity vector as the asteroid, and we can study the motion of the asteroid with the ellipsoid only.

5. Show that the ellipsoid and the plane are tangent at P at all times **(1.5 points)**.

   The ellipsoid and the plane are tangent if their normals at P coincide. A normal vector (not necessarily unitary) to the ellipsoid can be computed by taking the gradient of Eq. (5)

$$\boldsymbol{n} = x_a\boldsymbol{i}_a + 2y_a\boldsymbol{j}_a + 4z_a\boldsymbol{k}_a \tag{9}$$

   This vector is parallel to $\boldsymbol{H}_{G,0}$, which is the normal to the plane. Thus, the two objects are tangent at all times.

6. Using the results of previous questions, describe qualitatively the motion of the ellipsoid with respect to the plane **(1.5 points)**.

   The point P is the instantaneous point of contact between the plane and the ellipsoid, which can be regarded as two rigid bodies. The ellipsoid's center G and the plane are fixed in inertial space. The ellipsoid performs a pure rotation with the angular velocity $\boldsymbol{\omega}_{a0}$ of the asteroid about an instantaneous axis of rotation that contains $P(a)$, the point of the ellipsoid that instantaneously coincides with P. Since $P(a)$ is in the instantaneous axis of rotation of the ellipsoid, its velocity is zero, like the velocity of all points in the plane. Thus, the ellipsoid rolls (and spins) on the plane without slipping.

   Additional background: the ellipsoid is the *ellipsoid of inertia* introduced by Poinsot in his study of the torque-free motion of a rigid body. The path that P describes on the ellipsoid is named *polhode*, and the path it describes on the plane is called *herpolhode*.

HINT: An ellipsoid can be expressed as $x^2/a + y^2/b + z^2/c = d$ with $a, b, c, d > 0$.

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