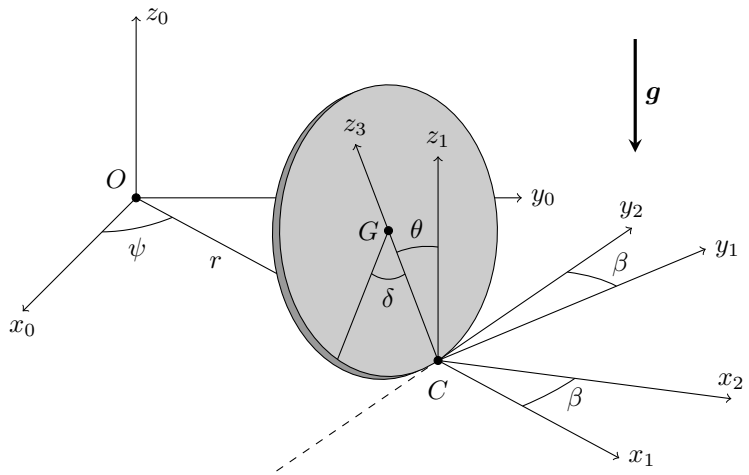
A heavy, homogeneous coin b of mass m, radius R and negligible thickness rolls on a horizontal plane at rest. The coin never takes off from the plane. We call C the instantaneous contact point between the coin and the plane, and G the center of mass of the coin. An inertial reference frame $S_0 : \{O; B_0\}$, with O a point of the plane and $B_0 : \{\boldsymbol{i}_0, \boldsymbol{j}_0, \boldsymbol{k}_0\}$, is given. We define r and ψ, which are the polar coordinates of C in the Oxy plane, and we call $S_1 : \{C; B_1\}$ the local reference frame associated to these polar coordinates. Lastly, we define β, θ, δ, which are 3-2-1 Euler angles of the coin with respect to S_1, positive in the direction shown in the figure (the coin is contained in the Cy_2z_3 plane).



1. How many *effective* degrees of freedom does the motion of the coin have, when the coin can roll and slip, and when the coin can roll but not slip? **(0.5 points)**.

   When the coin can roll and slip, the velocity of point $C(b)$ of the coin along $\boldsymbol{k}_0$ must be zero. This is a single scalar constraint, which reduces the number of degrees of the coin from 6 to 5.

   When the coin can roll but not slip, the three components of the velocity of $C(b)$ must be zero. These three scalar constraints reduce the number of degrees of freedom of the coin from 6 to 3.

2. Assuming that the coin can slip, obtain the general expression of $\boldsymbol{\omega}_{B0}$, $\boldsymbol{\alpha}_{B0}$, $\boldsymbol{r}_0^G$, $\boldsymbol{v}_0^G$, and $\boldsymbol{a}_0^G$ as a function of the coordinates of the problem and their time derivatives **(3.5 points)**.

   We begin by writing the definition of vector bases B_1, B_2 and B_3:

$$\begin{cases} \boldsymbol{i}_1 = \cos\psi\boldsymbol{i}_0 + \sin\psi\boldsymbol{j}_0 \\ \boldsymbol{j}_1 = -\sin\psi\boldsymbol{i}_0 + \cos\psi\boldsymbol{j}_0 \\ \boldsymbol{k}_1 = \boldsymbol{k}_0 \end{cases} \quad \begin{cases} \boldsymbol{i}_2 = \cos\beta\boldsymbol{i}_1 + \sin\beta\boldsymbol{j}_1 \\ \boldsymbol{j}_2 = -\sin\beta\boldsymbol{i}_1 + \cos\beta\boldsymbol{j}_1 \\ \boldsymbol{k}_2 = \boldsymbol{k}_1 \end{cases} \quad \begin{cases} \boldsymbol{i}_3 = \cos\theta\boldsymbol{i}_2 + \sin\theta\boldsymbol{k}_2 \\ \boldsymbol{j}_3 = \boldsymbol{j}_2 \\ \boldsymbol{k}_3 = -\sin\theta\boldsymbol{i}_2 + \cos\theta\boldsymbol{k}_2 \end{cases} \tag{1}$$

   For convenience, we call $\gamma = \beta + \psi$. The angular velocities relating each pair of consecutive frames, and the angular velocity and angular acceleration of the coin, are:

$$\boldsymbol{\omega}_{10} = \dot{\psi}\boldsymbol{k}_0; \quad \boldsymbol{\omega}_{21} = \dot{\beta}\boldsymbol{k}_0; \quad \boldsymbol{\omega}_{32} = -\dot{\theta}\boldsymbol{j}_2; \quad \boldsymbol{\omega}_{b3} = -\dot{\delta}\boldsymbol{i}_3; \tag{2}$$

$$\boldsymbol{\omega}_{b0} = -\dot{\delta}\boldsymbol{i}_3 - \dot{\theta}\boldsymbol{j}_2 + \dot{\gamma}\boldsymbol{k}_0 = [-\dot{\delta} + \dot{\gamma}\sin\theta]\boldsymbol{i}_3 - \dot{\theta}\boldsymbol{j}_3 + \dot{\gamma}\cos\theta\boldsymbol{k}_3$$
$$\boldsymbol{\alpha}_{b0} = [-\ddot{\delta} + \ddot{\gamma}\sin\theta + \dot{\theta}\dot{\gamma}\cos\theta]\boldsymbol{i}_3 + [-\dot{\delta}\dot{\gamma}\cos\theta - \ddot{\theta}]\boldsymbol{j}_3 + [-\dot{\delta}\dot{\theta} + \ddot{\gamma}\cos\theta - \dot{\theta}\dot{\gamma}\sin\theta]\boldsymbol{k}_3$$

   Note that, on addition to the expressions of $\boldsymbol{\omega}_{b0}$ and $\boldsymbol{\alpha}_{b0}$ being shorter in B_3, this basis is more convenient for writing the equation of moment of forces; it is not necessary to write the solution in B_0.

The position, velocity and acceleration of the center of mass are:

$$\boldsymbol{r}_0^G = r\boldsymbol{i}_1 + R\boldsymbol{k}_3$$
$$\boldsymbol{v}_0^G = \dot{r}\boldsymbol{i}_1 + r\dot{\psi}\boldsymbol{j}_1 - R\dot{\gamma}\sin\theta\boldsymbol{j}_2 - R\dot{\theta}\boldsymbol{i}_3$$
$$\boldsymbol{a}_0^G = [\ddot{r} - r\dot{\psi}^2]\boldsymbol{i}_1 + [2\dot{r}\dot{\psi} + r\ddot{\psi}]\boldsymbol{j}_1 + R\dot{\gamma}^2\sin\theta\boldsymbol{i}_2 - [R\ddot{\gamma}\sin\theta + 2R\dot{\theta}\dot{\gamma}\cos\theta]\boldsymbol{j}_2 - R\ddot{\theta}\boldsymbol{i}_3 - R\dot{\theta}^2\boldsymbol{k}_3$$

In this case, we observe that using mixed vector bases a set of short expressions can be obtained; it is not necessary to write the solution in B_0. The expression of $\boldsymbol{v}_0^G$ in B_3 is useful for the next question.

From this point on, consider that the friction coefficient between the coin and the plane is $\mu = \infty$, so the coin can roll but not slip:

3. Derive the scalar constraint equation(s) between the time derivatives of the coordinates of the problem that result from the no-slipping condition **(1.5 points)**.

   The constraint of rolling without slip in this case is $\boldsymbol{v}_0^{C(b)} = \boldsymbol{0}$. From the equation of the velocity field of the coin,

$$\begin{aligned}\boldsymbol{v}_0^G &= \boldsymbol{\omega}_{b0} \times \boldsymbol{CG} = R\boldsymbol{\omega}_{b0} \times \boldsymbol{k}_3 \\ &= -R\dot{\theta}\boldsymbol{i}_3 + R[\dot{\delta} - \dot{\gamma}\sin\theta]\boldsymbol{j}_3\end{aligned}$$

   Projecting this vector equation along B_3 we obtain:

$$\begin{aligned}(\dot{r}\cos\beta + r\dot{\psi}\sin\beta)\cos\theta &= 0 \\ -\dot{r}\sin\beta + r\dot{\psi}\cos\beta &= R\dot{\delta} \\ (\dot{r}\cos\beta + r\dot{\psi}\sin\beta)\sin\theta &= 0\end{aligned}$$

   The first and third equations provide the same information: the coin cannot slip along $\boldsymbol{i}_2$, and this establishes a relation between $\dot{r}$ and $\dot{\psi}$:

$$\dot{r}\cos\beta + r\dot{\psi}\sin\beta = 0. \tag{3}$$

   Combining this with the second equation we can write the relation between $\dot{\psi}$ and $\dot{\delta}$ as:

$$r\dot{\psi} = R\dot{\delta}\cos\beta \tag{4}$$

4. Identify and express all the external forces acting on the coin and the torque that they produce about G, introducing any necessary additional unknowns **(1 points)**.

   The only applied force is the coin weight, $\boldsymbol{F}_W = -mg\boldsymbol{k}_0$. There is the normal reaction force due to the plane at C, $\boldsymbol{N} = N\boldsymbol{k}_0$, and the friction reaction force at C, $\boldsymbol{F}_F = F_{Fx}\boldsymbol{i}_2 + F_{Fy}\boldsymbol{j}_2$. The variables N, F_{Fx}, F_{Fy} are unknowns that must be obtained as part of the solution.

   The toque about G due to weight is zero. The torques about G due to the normal and friction forces are:

$$\boldsymbol{M}_N = -RN\sin\theta\boldsymbol{j}_3$$
$$\boldsymbol{M}_{F_F} = RF_{Fy}\boldsymbol{i}_3 - RF_{Fx}\cos\theta\boldsymbol{j}_3$$

5. Write down the equations of motion of the coin in scalar form. Discuss whether any simple conservation law exists in the problem and justify your answer **(2 points)**.

   The vector equation of forces reads:

$$m\boldsymbol{a}_0^G = \boldsymbol{F}_W + \boldsymbol{N} + \boldsymbol{F}_F \tag{5}$$

Projecting along the vectors of B_2 we obtain:

$$\boldsymbol{i}_2 : [\ddot{r} - r\dot{\psi}^2]\cos\beta + [2\dot{r}\dot{\psi} + r\ddot{\psi}]\sin\beta - R\ddot{\theta}\cos\theta + R(\dot{\theta}^2 + \dot{\gamma}^2)\sin\theta = F_{Fx}/m \tag{6}$$

$$\boldsymbol{j}_2 : -[\ddot{r} - r\dot{\psi}^2]\sin\beta + [2\dot{r}\dot{\psi} + r\ddot{\psi}]\cos\beta - R\ddot{\gamma}\sin\theta - 2R\dot{\theta}\dot{\gamma}\cos\theta = F_{Fy}/m \tag{7}$$

$$\boldsymbol{k}_2 : -R\ddot{\theta}\sin\theta - R\dot{\theta}^2\cos\theta = N/m - g \tag{8}$$

The vector equation of moment of forces about G reads:

$$\left.\frac{\mathrm{d}\boldsymbol{H}_{G,0}}{\mathrm{d}t}\right|_0 = \boldsymbol{M}_N + \boldsymbol{M}_{F_F}. \tag{9}$$

We shall compute the left hand side in $S_g = \{G; B_3\}$, where the inertia tensor of the coin is constant:

$$\left.\frac{\mathrm{d}\boldsymbol{H}_{G,0}}{\mathrm{d}t}\right|_0 = \bar{\bar{\mathcal{I}}}_g \left.\frac{\mathrm{d}\boldsymbol{\omega}_{b0}}{\mathrm{d}t}\right|_3 + \boldsymbol{\omega}_{30}\times\bar{\bar{\mathcal{I}}}_g\boldsymbol{\omega}_{b0} = \bar{\bar{\mathcal{I}}}_g\boldsymbol{\alpha}_{b0} + \boldsymbol{\omega}_{b0}\times\bar{\bar{\mathcal{I}}}_g\boldsymbol{\omega}_{b0} = \frac{mR^2}{4}$$
$$\times\Big(2[-\ddot{\delta} + \ddot{\gamma}\sin\theta + \dot{\theta}\dot{\gamma}\cos\theta]\boldsymbol{i}_3 + [-2\dot{\delta}\dot{\gamma}\cos\theta - \ddot{\theta} + \dot{\gamma}^2\sin\theta\cos\theta]\boldsymbol{j}_3 + [-2\dot{\delta}\dot{\theta} + \ddot{\gamma}\cos\theta]\boldsymbol{k}_3\Big),$$

where we have used

$$\bar{\bar{\mathcal{I}}}_g = \frac{mR^2}{4}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{33}. \tag{10}$$

Projecting along the vectors of B_3 we obtain:

$$\boldsymbol{i}_3 : \frac{mR^2}{2}[-\ddot{\delta} + \ddot{\gamma}\sin\theta + \dot{\theta}\dot{\gamma}\cos\theta] = RF_{Fy} \tag{11}$$

$$\boldsymbol{j}_3 : \frac{mR^2}{4}[-2\dot{\delta}\dot{\gamma}\cos\theta - \ddot{\theta} + \dot{\gamma}^2\sin\theta\cos\theta] = -RN\sin\theta - RF_{Fx}\cos\theta \tag{12}$$

$$\boldsymbol{k}_3 : -2\dot{\delta}\dot{\theta} + \ddot{\gamma}\cos\theta = 0 \tag{13}$$

Conservation laws: the mechanical energy of the coin is conserved, since weight is conservative, and the reaction forces do not exert work (the body point of application $C(b)$ has zero velocity). Angular momentum is conserved in $\boldsymbol{k}_3$ direction.

6. Finally, consider the particular case in which the coin moves with $\beta = 0$ and $\boldsymbol{r}_0^G = h\boldsymbol{k}_0$, where $h < R$ is constant. Give the value of r, θ, and the reaction forces in this case. What is the relation between h and $\dot{\psi}$? Comment on this result **(1.5 points)**.

   Now $\gamma = \psi$. It is easy to see that $\cos\theta = h/R = \text{const}$ and $R\sin\theta = r$. Note that these two relations do not hold if $\beta \neq 0$. Due to $\boldsymbol{a}_0^G = 0$, we obtain

$$F_{Fx} = 0; \quad F_{Fy} = 0; \quad N = mg. \tag{14}$$

From Eq. (13), $\dot{\psi} = \text{const}$. Then, from Eq. (11) (or from the no-slip constraint), $\dot{\delta} = \text{const}$. Finally, from Eq. (12) one finds:

$$\dot{\psi}^2 = \frac{4g}{h}. \tag{15}$$

This expression reveals a curious behaviour of the coin, namely, that the lower h is, the faster the contact point C circles around the origin, and as $h \to 0$, $\dot{\psi}^2 \to \infty$. Likewise, since $\dot{\delta} = \dot{\psi}r/R$, $\dot{\delta} \to \infty$ too. The speed up of the wobbling and spinning of a disk as its energy is dissipated is a well known aspect of Euler's disk (see, for example this video).

NOTE: The moment of inertia of a disk of mass m and radius R about its axis of rotational symmetry is $mR^2/2$.