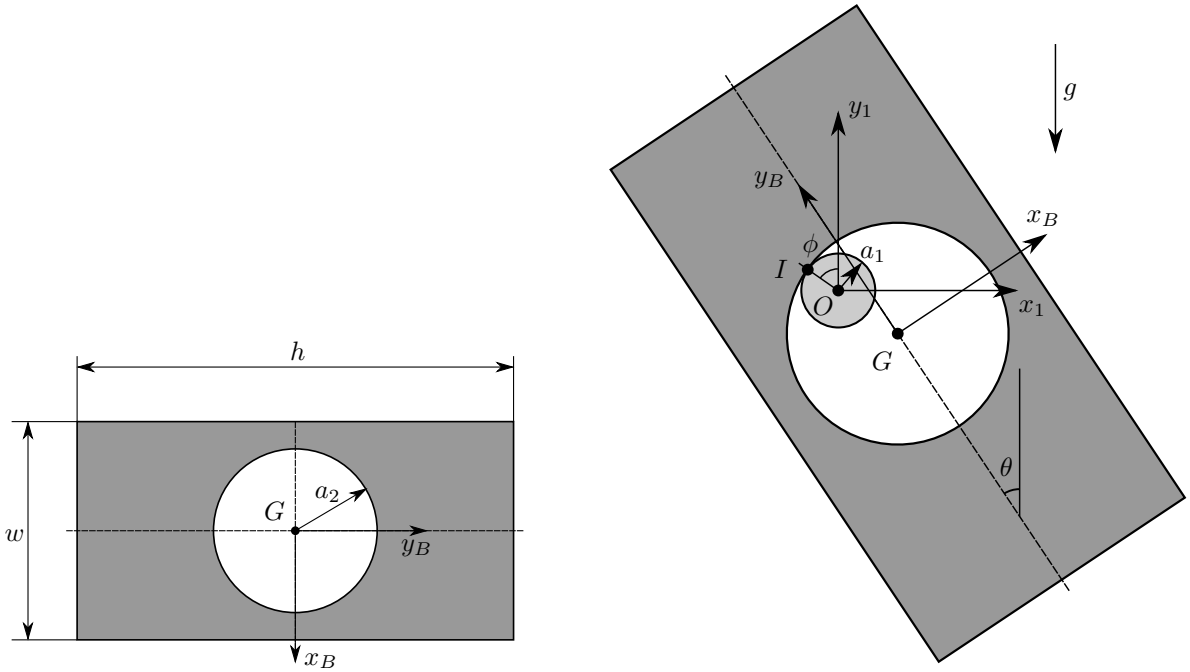
PROBLEM

10 points, 60 minutes.

A heavy rectangular piece of wood of mass m with a circular hole of radius a_2 is positioned on a fixed cylindrical knob of radius a_1 as shown below. We want to study the motion of the piece as a rigid body with respect to the inertial reference frame $Ox_1y_1z_1$ of the figure, assuming that the internal surface of the hole is always in contact with the knob at one point I and that the piece rolls without slip over the knob. We call θ the angle between the longitudinal symmetry axis of the piece and the vertical, and we call ϕ the angle between the vector OI and the vertical. Note that the two angles, θ and ϕ, do not need to coincide in general. The dimensions of the problem satisfy $h = 2w$, $a_2 = w/\sqrt{2\pi}$, $a_1 = a_2/3$.



1. Compute I_z, the moment of inertia of the piece along z_1 at G, as a function of m and w.

$$I_z = \left(\frac{5}{9} - \frac{1}{12\pi}\right) mw^2 \tag{1}$$

2. Write down the expressions of the velocity and acceleration of point G as a function of ϕ, θ and their time derivatives.

   We first define an intermediate set of axes $Ox_2y_2z_2$ such that $\boldsymbol{i}_2 = \cos\phi\boldsymbol{i}_1 + \sin\phi\boldsymbol{j}_1$ and $\boldsymbol{j}_2 = -\sin\phi\boldsymbol{i}_1 + \cos\phi\boldsymbol{j}_1$. We have:

$$\boldsymbol{r}_G = -\frac{2w}{3\sqrt{2\pi}}\boldsymbol{j}_2 = \frac{2w}{3\sqrt{2\pi}}\left(\sin\phi\boldsymbol{i}_1 - \cos\phi\boldsymbol{j}_1\right), \tag{2}$$

$$\boldsymbol{v}_G = \frac{2w}{3\sqrt{2\pi}}\dot{\phi}\left(\cos\phi\boldsymbol{i}_1 + \sin\phi\boldsymbol{j}_1\right), \tag{3}$$

$$\boldsymbol{a}_G = \frac{2w}{3\sqrt{2\pi}}\ddot{\phi}\left(\cos\phi\boldsymbol{i}_1 + \sin\phi\boldsymbol{j}_1\right) + \frac{2w}{3\sqrt{2\pi}}\dot{\phi}^2\left(-\sin\phi\boldsymbol{i}_1 + \cos\phi\boldsymbol{j}_1\right). \tag{4}$$

   Note that the answer could have also been written in terms of $\dot{\theta}$ with the solution of the next point in the problem. Note also that the angular velocity of the body is $\boldsymbol{\omega}_{B/1} = \dot{\theta}\boldsymbol{k}_1$.

3. Find the relation between $\dot{\phi}$ and $\dot{\theta}$ imposing the no-slip condition at I.

   The no-slip condition requires that the velocity of the point in contact with the knob at any instant be zero in the reference frame $Ox_1y_1z_1$, $\boldsymbol{v}_I = 0$.

$$\boldsymbol{v}_I = \boldsymbol{v}_G + \boldsymbol{\omega}_{B/1} \times GI = 0. \tag{5}$$

   Working with this condition one finds that

$$\dot{\theta} = \frac{2}{3}\dot{\phi}. \tag{6}$$

4. Write down the equations of motion that yield ϕ and the reaction forces as functions of time. Note that you do not need to solve these equations.

   The forces acting on the body are the weight $\boldsymbol{W} = -mg\boldsymbol{j}_1$ applied at G, and the normal and the friction $\boldsymbol{N} = N\boldsymbol{j}_2$, $\boldsymbol{F} = F\boldsymbol{i}_2$ applied at I. Newton equations along $\boldsymbol{i}_1$ and $\boldsymbol{j}_1$, and the equation of moments of forces along z_1 at G are:

$$m\frac{2w}{3\sqrt{2\pi}}\left(\ddot{\phi}\cos\phi - \dot{\phi}^2\sin\phi\right) = F\cos\phi - N\sin\phi \tag{7}$$

$$m\frac{2w}{3\sqrt{2\pi}}\left(\ddot{\phi}\sin\phi + \dot{\phi}^2\cos\phi\right) = -mg + F\sin\phi + N\cos\phi \tag{8}$$

$$\frac{2}{3}\left(\frac{5}{9} - \frac{1}{12\pi}\right)mw^2\ddot{\phi} = -F\frac{w}{\sqrt{2\pi}} \tag{9}$$

5. From these equations, find the second-order differential equation for the evolution of ϕ that does not depend on the reaction forces.

   Substituting F from Eq. (7) and (8) into Eq. (9):

$$\left(\frac{10}{27} - \frac{1}{18\pi} + \frac{1}{3\pi}\right)\ddot{\phi} + \frac{g}{w\sqrt{2\pi}}\sin(\phi) = 0 \tag{10}$$

$$\ddot{\phi} + \frac{27}{20\pi + 15}\frac{\sqrt{2\pi}g}{w}\sin(\phi) = 0 \tag{11}$$

   This is analogous to the equation of a pendulum.

6. If initially the wood piece is placed such that G and O are along the same vertical with $\theta_0 = 0$, the speed of G is $|\boldsymbol{v}_{G0}| = 2wA/3$ and the piece is rotating counterclockwise, find the initial conditions that should be imposed to integrate the equation from the previous question for ϕ.

   Equating with the general expression of $|\boldsymbol{v}_G|$ we have

$$\frac{2w}{3\sqrt{2\pi}}\dot{\phi}_0 = \frac{2wA}{3} \tag{12}$$

   So $\dot{\phi}_0 = A\sqrt{2\pi}$. Together with $\phi_0 = 0$, this is all we need to integrate Eq. (11). Note that the value of θ_0 is irrelevant for the integration of ϕ.

   (∗) *This problem was drafted with the help of python sympy classical mechanics module* (link). *If you want to play with this awesome piece of sofware, the easiest way is to install python anaconda distribution* (link).