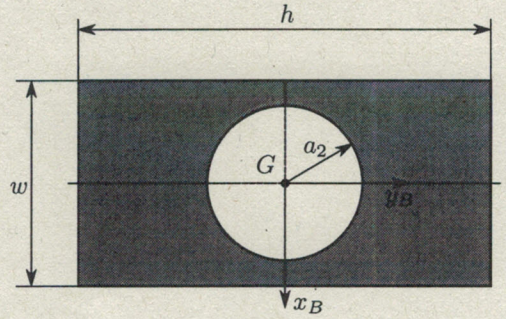
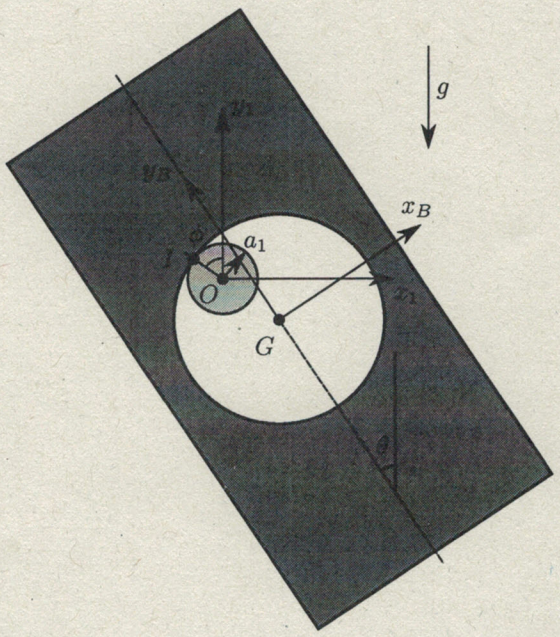
PROBLEM

10 points, 60 minutes.

A heavy rectangular piece of wood of mass m with a circular hole of radius a_2 is positioned on a fixed cylindrical knob of radius a_1 as shown below. We want to study the motion of the piece as a rigid body with respect to the inertial reference frame $Ox_1y_1z_1$ of the figure, assuming that the internal surface of the hole is always in contact with the knob at one point I and that the piece rolls without slip over the knob. We call θ the angle between the longitudinal symmetry axis of the piece and the vertical, and we call ϕ the angle between the vector OI and the vertical. Note that the two angles, θ and ϕ, do not need to coincide in general. The dimensions of the problem satisfy $h = 2w$, $a_2 = w/\sqrt{2\pi}$, $a_1 = a_2/3$.





1. Compute I_z, the moment of inertia of the piece along z_1 at G, as a function of m and w.
2. Write down the expressions of the velocity and acceleration of point G as a function of ϕ, θ and their time derivatives.
3. Find the relation between $\dot{\phi}$ and $\dot{\theta}$ imposing the no-slip condition at I.
4. Write down the equations of motion that yield ϕ and the reaction forces as functions of time. Note that you do not need to solve these equations.
5. From these equations, find the second-order differential equation for the evolution of ϕ that does not depend on the reaction forces.
6. If initially the wood piece is placed such that G and O are along the same vertical with $\theta_0 = 0$, the speed of G is $|\boldsymbol{v}_{G0}| = 2wA/3$ and the piece is rotating counterclockwise, find the initial conditions that should be imposed to integrate the equation from the previous question for ϕ.