We consider the plane motion in a vertical plane $0XY$ of a heavy equilateral triangle of mass m and side length a. The motion of the triangle is influenced by the spring of constant k which connects the origin of the inertial reference frame OXY to the vertex A of the triangle. In addition, the centre of mass of the triangle G is constrained to move along the vertical line $x = 2a$.

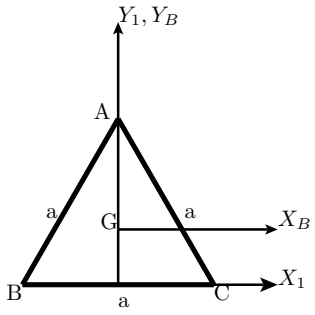
1. Determine the position of the centre of mass G with respect to the coordinate system X_1Y_1 shown below on the left. (1 point)

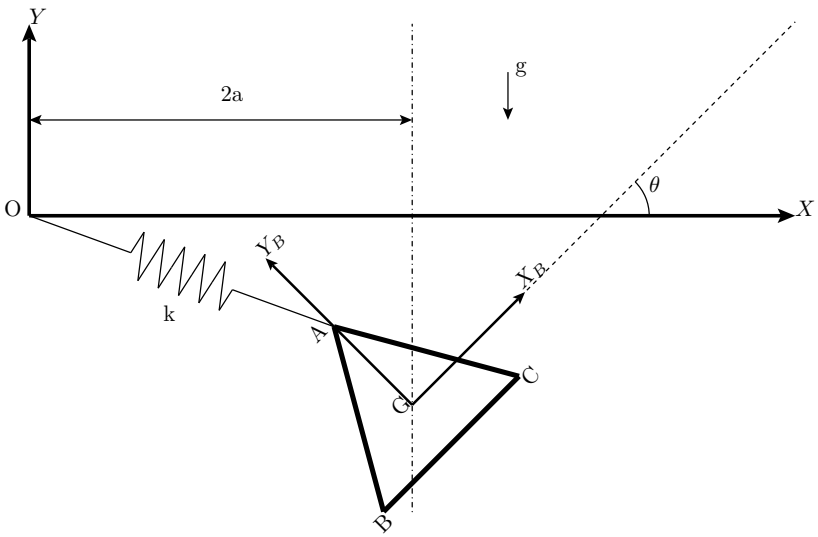
2. Determine the moment of inertia with respect to point G. (2 points)

3. If at time $t = 0$, the side BC of the triangle is located on the axis OX and the vertex A is on the line $x = 2a$ moving with velocity $\boldsymbol{v}_A = u_0\boldsymbol{i} + v_0\boldsymbol{j}$ (where $\boldsymbol{i}$,$\boldsymbol{j}$ are the unit vectors along the axes OX and OY).
   (a) Write down the equations of motion for the triangle. (2 points)
   (b) Obtain the required initial conditions to close the formulation of the problem. Determine also the instantaneous axis of rotation at $t = 0$. (1 point)

4. Now, we consider a different situation. The triangle is in the equilibrium position and at rest. The value of the angle is $\theta = \pi/4$. Determine the value of the spring constant k, the position of the centre of mass and the reaction force. (3 points)

5. In the equilibrium position computed before, we now fix the position of G and apply a small perturbation to the orientation of the triangle, such that $\theta = \pi/4 + \phi$ and $\phi \ll 1$. Determine the frequency of oscillation of the triangle with respect to the equilibrium position. (1 point)





1. The position of the centre of mass G can be determined in this case using symmetries. The axis Y_1 divides the triangle in two equal rectangular triangles, so G must lie along that axis. The line that joins vertex B and the mid point of segment AC divides also the triangle in two equal rectangular triangles, so G must also be located along that line. The intersection of those two lines is the position of the centre of mass. With respect to the coordinate system X_1Y_1 , the position of the centre of mass is:

$$\boldsymbol{r_G} = \frac{a}{2}\tan(30^\circ)\boldsymbol{j_1} = \frac{\sqrt{3}}{6}a\boldsymbol{j_1}. \tag{1}$$

Other quantities that will be used in the following are the height of the triangle $h = (\sqrt{3}/2)a$, the half-side $c = a/2$ and the distance from the center of mass G to the vertex A, $b = (\sqrt{3}/3)a$. The equation of the segment CA can be written as:

$$y = s(x - c), \tag{2}$$

with $s = -h/c$.

2. In order to compute the moment of inertia with respect to G, we are going to compute first the moment of inertia with respect to O_1 (mid-point of the segment BC). Then, we will use the parallel axis theorem to compute the moment of inertia with respect to G. Due to the symmetry of the problem, the moment of inertia of the triangle ABC with respect to O_1 is twice the moment of inertia of the triangle AO_1C with respect to O_1. The element of mass is

$$\mathrm{d}m = \frac{m}{hc}\mathrm{d}x\mathrm{d}y, \tag{3}$$

and the moment of inertia with respect to O_1

$$I_{O_1} = 2\int_0^c \int_0^{s(x-c)} (x^2 + y^2)\frac{m}{hc}\mathrm{d}x\mathrm{d}y = \frac{2m}{hc}\left[\int_0^c x^2\left(\int_0^{s(x-c)} \mathrm{d}y\right)\mathrm{d}x + \int_0^c \left(\int_0^{s(x-c)} y^2\mathrm{d}y\right)\mathrm{d}x\right]. \tag{4}$$

Integrating first along the y direction leads to

$$I_{O_1} = \frac{2m}{hc}\left[\int_0^c x^2 s(x-c)\mathrm{d}x + \frac{1}{3}\int_0^c (s(x-c))^3\mathrm{d}x\right] \tag{5}$$

and integrating along x we obtain

$$I_{O_1} = \frac{2}{3}mc^2 = \frac{1}{6}ma^2. \tag{6}$$

Finally, applying the parallel axis theorem

$$I_{O_1} = I_G + m|O_1G|^2 \rightarrow I_G = \frac{1}{12}ma^2. \tag{7}$$

Recall that the moment of inertia with respect to G for a plane body is equal to the moment of inertia with respect to an axis passing through G and perpendicular to the plane of the motion I_z. This is the only component of the tensor of inertia which is needed in the following.

3. In a general case, a rigid body has 6 degrees of freedom. In plane motion, they get reduced to 3. They are for example the two coordinates of the centre of mass and one angle of orientation of the body. In the present case due to the constraint there are only two degrees of freedom. They are the vertical coordinate of the centre of mass y_G and the angle θ that the GX_B axis forms with the OX axis. The position of the centre of mass with respect to the inertial reference frame OXY is

$$\boldsymbol{r_G} = 2a\boldsymbol{i} + y_G\boldsymbol{j}, \tag{8}$$

and therefore the velocity and acceleration of G are

$$\boldsymbol{v_G} = \dot{y}_G\boldsymbol{j}, \boldsymbol{a_G} = \ddot{y}_G\boldsymbol{j}. \tag{9}$$

The angular velocity of the body is given by $\boldsymbol{\omega} = \dot{\theta}\boldsymbol{k}$. The unit vectors along the axes GX_B and GY_B can be expressed with respect to the unit vectors $\boldsymbol{i}$ and $\boldsymbol{j}$

$$\boldsymbol{i_B} = \cos(\theta)\boldsymbol{i} + \sin(\theta)\boldsymbol{j}, \tag{10}$$
$$\boldsymbol{j_B} = -\sin(\theta)\boldsymbol{i} + \cos(\theta)\boldsymbol{j}. \tag{11}$$

(a) The general equations of motion are

$$\sum \boldsymbol{F} = m\boldsymbol{a_G} \tag{12}$$
$$\sum \boldsymbol{M_G} = \frac{\mathrm{d}\boldsymbol{H_G}}{\mathrm{d}t} \tag{13}$$

where $\sum \boldsymbol{F}$ is the sum of the forces applied to the triangle, $\sum \boldsymbol{M_G}$ is the sum of the torques applied to the triangle and $\boldsymbol{H_G}$ is the angular momentum with respect to G.
There are three forces applied to the triangle. The weight of the triangle, $\boldsymbol{F_W} = -mg\boldsymbol{j}$, the reaction acting on G, $\boldsymbol{F_R} = R\boldsymbol{i}$ and the force of the spring, $\boldsymbol{F_S} = -k\boldsymbol{OA}$. The vector $\boldsymbol{OA}$ is

$$\boldsymbol{OA} = \boldsymbol{OG} + \boldsymbol{GA} = 2a\boldsymbol{i} + y_G\boldsymbol{j} + b\boldsymbol{j_B} = (2a - b\sin(\theta))\boldsymbol{i} + (y_G + b\cos(\theta))\boldsymbol{j} \tag{14}$$

The vector equation (12) leads to the two scalar equations:

$$R + k(b\sin(\theta) - 2a) = 0, \tag{15}$$
$$-k(y_G + b\cos(\theta)) - mg = m\ddot{y}_G. \tag{16}$$

The weight and the reaction act on G and, therefore, they do not produce torque. The only torque is then due to the spring force:

$$M_G = \boldsymbol{GA} \times \boldsymbol{F_S} = -k\boldsymbol{GA} \times (\boldsymbol{OG} + \boldsymbol{GA}) = -k\boldsymbol{GA} \times \boldsymbol{OG} = kb(y_G\sin(\theta) + 2a\cos(\theta))\boldsymbol{k} \tag{17}$$

The angular momentum with respecto to G is $H_G = I_z\dot{\theta}\boldsymbol{k}$ and $\mathrm{d}H_G/\mathrm{d}t = I_z\ddot{\theta}\boldsymbol{k}$. Eq. (13) leads to

$$kb(y_G\sin(\theta) + 2a\cos(\theta)) = I_z\ddot{\theta}. \tag{18}$$

The equations (16) and (18) are a system of two coupled ordinary differential equations of second order for the two unknowns y_G and θ. The equation (15) would provide the value of the unknown reaction R once the value of θ is known.

(b) In order to solve equations (16) and (18) it is necessary to specify 4 initial conditions: the values at $t = 0$ of θ, $\dot{\theta}$, y_G and $\dot{y}_G$. There are two initial positions of the triangle that fulfill the conditions given in the exercise: vertex A can be located either at $(2a, h)$ or at $(2a, -h)$. In the following, we are going to consider only the first of them, but the exercise would be equally valid considering the second one (note that in that case the motion would be different). If vertex A is located at $(2a, h)$ then at $t = 0$, $y_G = \sqrt{3}a/6$ and $\theta = 0$. In order to get the velocities, consider the equation of the velocity field of the rigid body

$$\boldsymbol{v_A} = \boldsymbol{v_G} + \boldsymbol{\omega} \times \boldsymbol{GA} \tag{19}$$
$$u_0\boldsymbol{i} + v_0\boldsymbol{j} = \dot{y}_G\boldsymbol{j} + \dot{\theta}\boldsymbol{k} \times b\boldsymbol{j} \tag{20}$$
$$u_0\boldsymbol{i} + v_0\boldsymbol{j} = -b\dot{\theta}\boldsymbol{i} + \dot{y}_G\boldsymbol{j}. \tag{21}$$

Thus, at $t = 0$, $\dot{y}_G = v_0$ and $\dot{\theta} = -u_0/b$.

Now, the instantaneous axis of rotation in plane motion is perpendicular to the plane. Therefore, we only need to find the point P of the plane whose velocity is zero. Applying again the equation of the velocity field of the rigid body

$$\boldsymbol{v_P} = \boldsymbol{v_G} + \boldsymbol{\omega} \times \boldsymbol{GP} = 0 \tag{22}$$

$$0 = v_0\boldsymbol{j} - (u_0/b)\boldsymbol{k} \times [(x_P - 2a)\boldsymbol{i} + (y_P - \sqrt{3}a/6)\boldsymbol{j}] \tag{23}$$

and solving this vector equation we obtain:

$$x_P = 2a + b\frac{v_0}{u_0} \tag{24}$$

$$y_P = \sqrt{3}a/6 \tag{25}$$

4. If the triangle is in equilibrium, the sum of the forces and the sum of the torques applied to the triangle must be zero. In the equations (16) and (18) we must set the right hand side to zero. The equation (15) still holds.

   From equation (18) we get

$$y_G \sin\theta + 2a\cos\theta = 0 \rightarrow y_G = -2a\frac{\cos(\theta)}{\sin\theta} = -2a, \tag{26}$$

   since $\theta = \pi/4$. From equation (16) we get

$$k(y_G + b\cos\theta) + mg = 0 \rightarrow k = \frac{1}{\left(2 - \frac{\sqrt{6}}{6}\right)}\frac{mg}{a}. \tag{27}$$

   From equation (15) we get

$$R = ka\left(2 - \frac{\sqrt{6}}{6}\right) = mg. \tag{28}$$

5. We now fix the position of G, we can ignore equation (16). The equation of motion is (18). The angle $\theta = \pi/4 + \phi$, so $\ddot{\theta} = \ddot{\phi}$. The equation (18) can be rewritten

$$kb(y_G \sin(\pi/4 + \phi) + 2a\cos(\pi/4 + \phi)) = I_z\ddot{\phi}. \tag{29}$$

   Introducing $y_G = -2a$ and the formulas

$$\sin(\pi/4 + \phi) = \sin(\pi/4)\cos(\phi) + \cos(\pi/4)\sin(\phi) \tag{30}$$

$$\cos(\pi/4 + \phi) = \cos(\pi/4)\cos(\phi) - \sin(\pi/4)\sin(\phi), \tag{31}$$

   we obtain

$$-2\sqrt{2}abk\sin(\phi) = I_z\ddot{\phi}. \tag{32}$$

   Using the fact that $\phi \ll 1$, we can use as a first approximation $\sin(\phi) \simeq \phi$. The final equation we obtain is:

$$\ddot{\phi} + \frac{2\sqrt{2}abk}{I_z}\phi = 0, \tag{33}$$

   which is the equation of an harmonic oscillator of natural frequency squared

$$\Omega^2 = \frac{2\sqrt{2}abk}{I_z} = 8\sqrt{6}\frac{k}{m}, \tag{34}$$

   in the last expression we have introduced the values of b and $I_z = I_G$ obtained above.