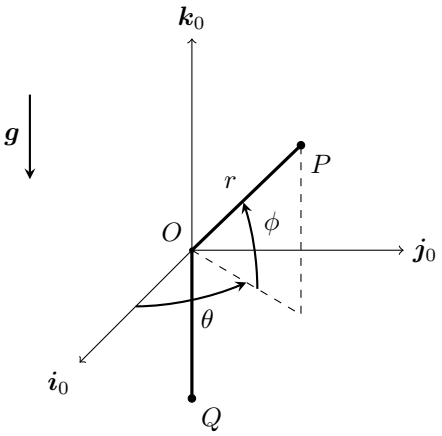
Two heavy particles P (mass m) and Q (**caution:** mass $2m$) are connected by an inextensible, massless string of length ℓ. The string goes through a tiny ring located at the origin O of the S_0 reference frame, as shown in the figure. There is no friction at the ring. Particle P can move in any direction, whereas Q can only move up and down along the vertical axis. Assume that the string is always in tension and that the particles, in their motion, never reach the origin O. We introduce the spherical coordinates $\{r, \theta, \phi\}$ for the position of P as shown in the figure.



1. Give the expressions for the position, velocity, and acceleration vectors of P and Q with respect to S_0, as functions of r, θ, ϕ and their derivatives. Remember that you may use any unit vectors in your solution, provided that you define them first (**2 points**).

   We first define the following unit vectors

$$\begin{cases} \boldsymbol{e}_r = \cos\phi\cos\theta\boldsymbol{i}_0 + \cos\phi\sin\theta\boldsymbol{j}_0 + \sin\phi\boldsymbol{k}_0, \\ \boldsymbol{e}_\theta = -\sin\theta\boldsymbol{i}_0 + \cos\theta\boldsymbol{j}_0, \\ \boldsymbol{e}_\phi = -\sin\phi\cos\theta\boldsymbol{i}_0 - \sin\phi\sin\theta\boldsymbol{j}_0 + \cos\phi\boldsymbol{k}_0, \end{cases} \tag{0.1}$$

   Then, the solution can be written as:

$$\begin{aligned} \boldsymbol{r}_0^P &= r\boldsymbol{e}_r, \\ \boldsymbol{v}_0^P &= \dot{r}\boldsymbol{e}_r + r\dot{\theta}\cos\phi\boldsymbol{e}_\theta + r\dot{\phi}\boldsymbol{e}_\phi, \\ \boldsymbol{a}_0^P &= (\ddot{r} - r\dot{\theta}^2\cos^2\phi - r\dot{\phi}^2)\boldsymbol{e}_r \\ &\quad + (2\dot{r}\dot{\theta}\cos\phi + r\ddot{\theta}\cos\phi - 2r\dot{\theta}\dot{\phi}\sin\phi)\boldsymbol{e}_\theta \\ &\quad + (2\dot{r}\dot{\phi} + r\ddot{\phi} + r\dot{\theta}^2\cos\phi\sin\phi)\boldsymbol{e}_\phi, \end{aligned} \qquad \begin{aligned} \boldsymbol{r}_0^Q &= -(\ell - r)\boldsymbol{k}_0, \\ \boldsymbol{v}_0^Q &= \dot{r}\boldsymbol{k}_0, \\ \boldsymbol{a}_0^Q &= \ddot{r}\boldsymbol{k}_0. \end{aligned}$$

2. Identify and write down all forces on P and Q (**2 points**).

   We have the following forces on P:

$$\begin{aligned} \boldsymbol{W}_P &= -mg\boldsymbol{k}_0 \\ \boldsymbol{T}_P &= -T\boldsymbol{e}_r \end{aligned}$$

   and the following forces on Q:

$$\begin{aligned} \boldsymbol{W}_Q &= -2mg\boldsymbol{k}_0 \\ \boldsymbol{T}_Q &= T\boldsymbol{k}_0 \end{aligned}$$

   where T is an unknown.

3. Write down three differential equations for r, θ, ϕ, free of any other unknowns, and an expression for the constraint unknown. The latter expression may not depend on the second derivatives of the coordinates (**3 points**).

   Projecting the vector equation of motion in S_0 along $\boldsymbol{e}_r$, $\boldsymbol{e}_\theta$ and $\boldsymbol{e}_\phi$:

$$\begin{aligned} m(\ddot{r} - r\dot{\theta}^2\cos^2\phi - r\dot{\phi}^2) &= -mg\sin\phi - T \\ m(2\dot{r}\dot{\theta}\cos\phi + r\ddot{\theta}\cos\phi - 2r\dot{\theta}\dot{\phi}\sin\phi) &= 0 \\ m(2\dot{r}\dot{\phi} + r\ddot{\phi} + r\dot{\theta}^2\cos\phi\sin\phi) &= -mg\cos\phi. \end{aligned}$$

   Then, the equation of motion of Q along $\boldsymbol{k}_0$ yields:

$$2m\ddot{r} = -2mg + T.$$

   Eliminating T, we find the equation for r:

$$m(3\ddot{r} - r\dot{\theta}^2\cos^2\phi - r\dot{\phi}^2) = -mg(2 + \sin\phi).$$

   Eliminating $\ddot{r}$, we find the equation for T:

$$T = \frac{2m}{3}\left[g(1 - \sin\phi) + r\dot{\theta}^2\cos^2\phi + r\dot{\phi}^2\right].$$

4. Figure out the value of ϕ and $r\dot{\theta}^2$ for which the particle P moves in horizontal circles (**1 point**).

   Forcing that $\dot{\phi}, \ddot{\phi}, \dot{r}, \ddot{r}$ be zero in the equations above:

$$\begin{aligned} r\dot{\theta}^2\sin\phi &= -g, \\ r\dot{\theta}^2\cos^2\phi &= g(2 + \sin\phi). \end{aligned}$$

   There is only one value of ϕ that allows this solution, satisfying the equation:

$$0 = 2\sin\phi + \sin^2\phi + \cos^2\phi \qquad \Rightarrow \qquad 0 = 2\sin\phi + 1$$

   The solution is only valid for $\phi = -30$ deg. Hence, $r\dot{\theta}^2 = 2g$.

5. Write down two first integrals of motion of the system (**2 points**).

   The angular momentum of P about O along $\boldsymbol{k}_0$ is conserved. This can also be found by integrating the θ equation (multiply by the integrating factor $r\cos\phi$ first):

$$r^2\dot{\theta}\cos^2\phi = \text{const}$$

   The mechanical energy of the system is conserved, as the weight forces are conservative and the tension only exchanges energy between P and Q:

$$\frac{1}{2}m\left[\dot{r}^2 + r^2\dot{\theta}^2\cos^2\phi + r^2\dot{\phi}^2\right] + m\dot{r}^2 + mgr(2 + \sin\phi) = \text{const}$$