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nsabenocontesta

Mecánica aplicada a la Ingeniería Aeroespacial

2º Grado en Ingeniería Aeroespacial

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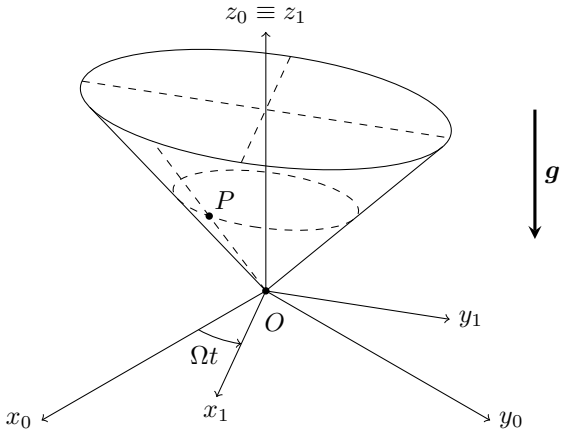
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A heavy point particle P with mass m moves without friction on a cone of elliptic cross-section as shown in the figure. We define an auxiliary reference frame $S_1 : \{O; B_1\}$ attached to the cone. The cone is defined in term of two parameters ζ, β as

$$\text{Elliptic cone:} \quad \begin{cases} x_1 = \zeta \cos\beta \\ y_1 = 2\zeta \sin\beta \\ z_1 = \zeta, \end{cases} \tag{1}$$

The cone spins about its Oz_1 axis at a constant angular rate Ω with respect to the inertial reference frame $S_0 : \{O; B_0\}$. At time $t = 0$, the reference frame S_1 coincides with S_0, the particle P is located at $\boldsymbol{r}_{1,0}^P = 2\boldsymbol{j}_1 + \boldsymbol{k}_1$, and has a velocity $\boldsymbol{v}_{1,0}^P = \boldsymbol{i}_1$.



1. Obtain the general expression for the position, velocity, and acceleration of P **in reference frame** S_1 as a function of ζ, β and their derivatives. Remember to define any unit vectors that you use in terms of the vectors of B_0 **(2 points)**.

   We define the auxiliary vector basis B_1:

   $$\begin{cases} \boldsymbol{i}_1 = \cos(\Omega t)\boldsymbol{i}_0 + \sin(\Omega t)\boldsymbol{j}_0, \\ \boldsymbol{j}_1 = -\sin(\Omega t)\boldsymbol{i}_0 + \cos(\Omega t)\boldsymbol{j}_0, \\ \boldsymbol{k}_1 = \boldsymbol{k}_0, \end{cases} \tag{2}$$

   The requested expressions are:

   $$\begin{aligned} \boldsymbol{r}_1^P &= \zeta\cos\beta\boldsymbol{i}_1 + 2\zeta\sin\beta\boldsymbol{j}_1 + \zeta\boldsymbol{k}_1 \\ \boldsymbol{v}_1^P &= (\dot\zeta\cos\beta - \zeta\dot\beta\sin\beta)\boldsymbol{i}_1 + 2(\dot\zeta\sin\beta + \zeta\dot\beta\cos\beta)\boldsymbol{j}_1 + \dot\zeta\boldsymbol{k}_1 \\ \boldsymbol{a}_1^P &= (\ddot\zeta\cos\beta - 2\dot\zeta\dot\beta\sin\beta - \zeta\ddot\beta\sin\beta - \zeta\dot\beta^2\cos\beta)\boldsymbol{i}_1 \\ &\quad + 2(\ddot\zeta\sin\beta + 2\dot\zeta\dot\beta\cos\beta + \zeta\ddot\beta\cos\beta - \zeta\dot\beta^2\sin\beta)\boldsymbol{j}_1 \\ &\quad + \ddot\zeta\boldsymbol{k}_1 \end{aligned}$$

2. Identify and express mathematically all the forces acting on P as seen from S_1 **(3 points)**.

   The actual forces on the particle are its weight and the normal reaction of the cone. The implicit equation for the cone is $f(x_1, y_1, z_1) = 2x_1^2 + y_1^2/2 - 2z_1^2 = 0$, and taking the gradient we find a normal vector to the surface: $\nabla f = 4\zeta\cos\beta\boldsymbol{i}_1 + 2\zeta\sin\beta\boldsymbol{j}_1 - 4\zeta\boldsymbol{k}_1$. The two forces can be expressed as:

   $$\begin{aligned} \boldsymbol{W} &= -mg\boldsymbol{k}_1 \\ \boldsymbol{N} &= 2\lambda\cos\beta\boldsymbol{i}_1 + \lambda\sin\beta\boldsymbol{j}_1 - 2\lambda\boldsymbol{k}_1 \end{aligned}$$

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where λ is an unknown related to the magnitude of $\boldsymbol{N}$ to be computed as part of the solution. Additionally, since we are looking at the motion of P from a rotating reference frame, centrifugal and Coriolis inertial forces are present:

$$\boldsymbol{F}_{cf} = m\Omega^2(\zeta\cos\beta\boldsymbol{i}_1 + 2\zeta\sin\beta\boldsymbol{j}_1)$$
$$\boldsymbol{F}_{Cor} = 2m\Omega[2(\dot\zeta\sin\beta + \zeta\dot\beta\cos\beta)\boldsymbol{i}_1 + (-\dot\zeta\cos\beta + \zeta\dot\beta\sin\beta)\boldsymbol{j}_1]$$

3. Obtain two second-order differential equations for the evolution of ζ, β in time, which do not depend on any other unknowns in the problem **(2 points)**.

   The unknowns of the problem are ζ, β, and λ. After projecting, Newton's second law provides three scalar equations of motion:

   $$m\boldsymbol{a}_1^P = \boldsymbol{W} + \boldsymbol{N} + \boldsymbol{F}_{cf} + \boldsymbol{F}_{Cor}.$$

   To avoid λ in two of the scalar equations, we project this vector equation onto two directions perpendicular to $\boldsymbol{N}$. These two directions are easily obtained by differentiating $\boldsymbol{r}_1^P$ with respect to ζ, β:

   $$\boldsymbol{e}_\zeta = \cos\beta\boldsymbol{i}_1 + 2\sin\beta\boldsymbol{j}_1 + \boldsymbol{k}_1$$
   $$\boldsymbol{e}_\beta = -\sin\beta\boldsymbol{i}_1 + 2\cos\beta\boldsymbol{j}_1$$

   (note that these vectors are not unitary, not orthogonal, and that we have divided by ζ in the second one). The two equations sought are:

   $$\boldsymbol{e}_\zeta: \quad \ddot\zeta + (\ddot\zeta - \zeta\dot\beta^2)(1+3\sin^2\beta) + 3\left(2\dot\zeta\dot\beta + \zeta\ddot\beta\right)\cos\beta\sin\beta = -g + \Omega^2\zeta(1+3\sin^2\beta) + 4\Omega\zeta\dot\beta$$
   $$\boldsymbol{e}_\beta: \quad 3(\ddot\zeta - \zeta\dot\beta^2)\cos\beta\sin\beta + (2\dot\zeta\dot\beta + \zeta\ddot\beta)(1+3\cos^2\beta) = 3\Omega^2\zeta\cos\beta\sin\beta - 4\Omega\dot\zeta$$

   Another way of obtaining these expressions is to project in any vector basis, then find two linear combinations that do not depend on the reaction force. When solved, these two equations yield the evolution of ζ, β in time. Once these are known, the remaining scalar equation (along $\boldsymbol{N}$) would yield λ as a function of time.

4. Find, justify, and write down one conservation law in this problem, including the value of any constants there might be **(2 points)**.

   In S_1, the only forces that exert work on P are $\boldsymbol{W}$ and $\boldsymbol{F}_{cf}$, and both are conservative forces that derive from a potential:

   $$V_g = mg\zeta,$$
   $$V_{cf} = -\frac{1}{2}m\Omega^2\zeta^2(1+3\sin^2\beta).$$

   Thus, the mechanical energy of P in S_1 is a conserved quantity of motion, $E_1 = T_1 + V_g + V_{cf}$, with

   $$T_1 = \frac{1}{2}m[(\dot\zeta\cos\beta - \zeta\dot\beta\sin\beta)^2 + 4(\dot\zeta\sin\beta + \zeta\dot\beta\cos\beta)^2 + \dot\zeta^2]$$

   The initial values of the position and velocity of P allow computing E_1:

   $$\zeta_0 = 1; \quad \dot\zeta_0 = 0; \quad \beta_0 = \pi/2; \quad \dot\beta_0 = -1, \tag{3}$$

   from which:

   $$E_1 = m\left(\frac{1}{2} + g - 2\Omega^2\right). \tag{4}$$

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5. Find the equilibrium positions for P in S_1 for $\zeta > 0$ as a function of the constants of the problem **(1 point)**.

   Equating to 0 all time derivatives in the equations of motion along $\boldsymbol{e}_\zeta$ and $\boldsymbol{e}_\beta$ we obtain the conditions that ζ, β must satisfy for equilibrium:

$$0 = -g + \Omega^2 \zeta (1 + 3\sin^2 \beta)$$
$$0 = \cos\beta \sin\beta$$

   From the second equation, we see that only the positions with $\beta = \pi k/2$ with $k \in \mathbb{Z}$ can be equilibrium positions. For $\beta = \pi k$, the condition on ζ is $\zeta = g/\Omega^2$, while for $\beta = \pi k + \pi/2$ the condition is $\zeta = g/(4\Omega^2)$.

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