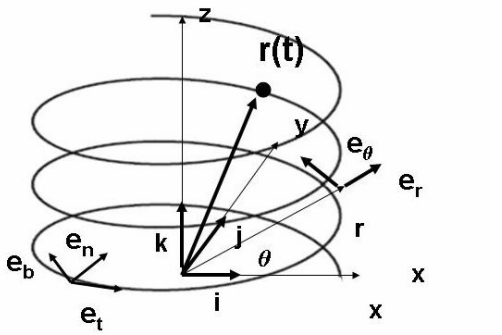
An out-of-control rocket traces an upward helical trajectory described in cartesian coordinates for $t > 0$ by

$$x = R\cos(\omega t), \tag{1}$$
$$y = R\sin(\omega t), \tag{2}$$
$$z = v_z t. \tag{3}$$

where v_z and ω are constants.

1. Express the position, velocity and acceleration vectors in the cartesian x, y, z system using unit vectors $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$.
2. Express the position, velocity and acceleration vectors in the cylindrical coordinate system using unit vectors $\boldsymbol{e}_R, \boldsymbol{e}_\theta, \boldsymbol{e}_z$.
3. Express the velocity and acceleration vectors using the intrinsic s, ρ coordinates and vector basis $\boldsymbol{e}_t, \boldsymbol{e}_n, \boldsymbol{e}_b$.
4. Compute the modulus of the velocity vector
5. Express the evolution of the arc-length coordinate as a function of time
6. Write down the trajectory as a function of the arc-length coordinate
7. Where is the center of curvature for this curve when the particle is located at $(x, y, z) = (R, 0, 0)$?



(Taken from MIT OCW)

Full Solution

1. *Position, velocity and acceleration can be expressed in cartesian basis as follows:*

$$\boldsymbol{r}_0^P = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k} = R\cos(\omega t)\boldsymbol{i} + R\sin(\omega t)\boldsymbol{j} + v_z t\boldsymbol{k}. \tag{4}$$
$$\boldsymbol{v}_0^P = \frac{d\boldsymbol{r}}{dt} = -R\omega\sin(\omega t)\boldsymbol{i} + R\omega\cos(\omega t)\boldsymbol{j} + v_z\boldsymbol{k}. \tag{5}$$
$$\boldsymbol{a}_0^P = \frac{d^2\boldsymbol{r}}{dt^2} = \frac{d\boldsymbol{v}}{dt} = -R\omega^2\cos(\omega t)\boldsymbol{i} - R\omega^2\sin(\omega t)\boldsymbol{j}. \tag{6}$$

2. *First of all we need to define the vector basis $\boldsymbol{e}_R, \boldsymbol{e}_\theta, \boldsymbol{e}_z$. The first two are the vectors from polar coordinates. In this case they can be expressed as:*

$$\boldsymbol{e}_R = \cos(\omega t)\boldsymbol{i} + \sin(\omega t)\boldsymbol{j}; \quad \boldsymbol{e}_\theta = -\sin(\omega t)\boldsymbol{i} + \cos(\omega t)\boldsymbol{j}; \quad \boldsymbol{e}_z = \boldsymbol{k}. \tag{7}$$

The position vector can then be obtained:

$$\boldsymbol{r}_0^P = R\boldsymbol{e}_R + v_z t\boldsymbol{k}. \tag{8}$$

Differentiating with respect to time, one finds:

$$\boldsymbol{v}_0^P = R\frac{d(\boldsymbol{e}_R)}{dt} + v_z\boldsymbol{k} = R\omega\boldsymbol{e}_\theta + v_z\boldsymbol{k}. \tag{9}$$
$$\boldsymbol{a}_0^P = R\omega\frac{d\boldsymbol{e}_\theta}{dt} = -R\omega^2\boldsymbol{e}_R. \tag{10}$$

3. *It can be seen that as* $\boldsymbol{v}_0^P \cdot \boldsymbol{a}_0^P = 0$*, there is no tangential acceleration so that one can write:*

$$\boldsymbol{v}_0^P = v\boldsymbol{e}_t; \quad \boldsymbol{a}_0^P = \gamma_n \boldsymbol{e}_n, \tag{11}$$

where

$$\boldsymbol{e}_t = \frac{\boldsymbol{v}_0^P}{v} = \frac{R\omega\boldsymbol{e}_\theta + v_z\boldsymbol{k}}{\sqrt{R^2\omega^2 + v_z^2}}, \quad \boldsymbol{e}_n = -\boldsymbol{e}_R; \quad \boldsymbol{e}_b = \boldsymbol{e}_t \times \boldsymbol{e}_n = \frac{-v_z\boldsymbol{e}_\theta + R\omega\boldsymbol{k}}{\sqrt{R^2\omega^2 + v_z^2}} \tag{12}$$

The velocity and acceleration can thus be written as:

$$\boldsymbol{v}_0^P = \sqrt{R^2\omega^2 + v_z^2}\boldsymbol{e}_t; \quad \boldsymbol{a}_0^P = R\omega^2\boldsymbol{e}_n. \tag{13}$$

Thus the radius of curvature is found to be:

$$\rho = \frac{v^2}{\gamma_n} = \frac{R^2\omega^2 + v_z^2}{R\omega^2} = R + \frac{v_z^2}{R\omega^2} \tag{14}$$

4. *The modulus of the velocity vector is* $\dot{s} = v = \sqrt{R^2\omega^2 + v_z^2}$*.*

5.

$$s(t) = \int_0^t \dot{s}dt = \int_0^t \sqrt{R^2\omega^2 + v_z^2}dt = \sqrt{R^2\omega^2 + v_z^2}t. \tag{15}$$

6. *Substituting* t *by* s *in the trajectory:*

$$x = R\cos\left(\frac{\omega s}{\sqrt{R^2\omega^2 + v_z^2}}\right) \tag{16}$$

$$y = R\sin\left(\frac{\omega s}{\sqrt{R^2\omega^2 + v_z^2}}\right) \tag{17}$$

$$z = v_z\frac{s}{\sqrt{R^2\omega^2 + v_z^2}} \tag{18}$$

7. *At* $(x, y, z) = (R, 0, 0)$*,* $t = 0$ *and therefore* $s = 0$*. The center of curvature is at a distance* ρ *in the* $\boldsymbol{e}_n$ *direction which for this case is* $-\boldsymbol{e}_r$*, and for these initial conditions* $\boldsymbol{e}_n(t = 0) = -\boldsymbol{i}$*. Thus position of the center of curvature is:*

$$\boldsymbol{r}_{CoC}(t = 0) = (R, 0, 0) - \rho\boldsymbol{i} = \left(-\frac{v_z^2}{R\omega^2}, 0, 0\right). \tag{19}$$

Just for your information, note that the center of curvature trajectory is also a helix rotating exactly in the same direction as the rocket with radius $\frac{v_z^2}{R\omega^2}$ *starting of from this initial location.*