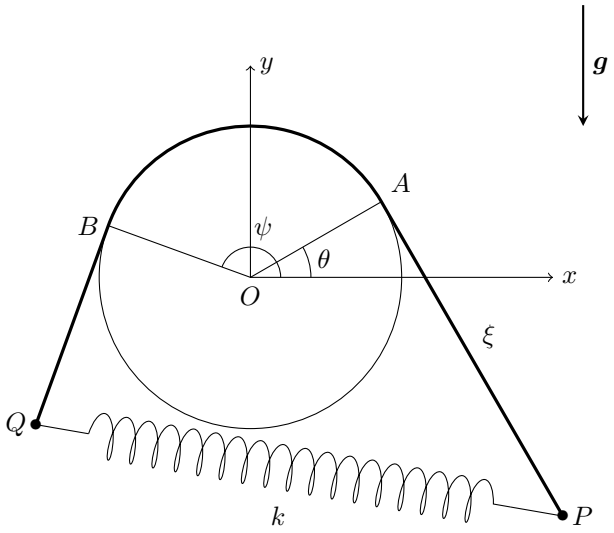
Two heavy particles P and Q of mass m each are connected by a flexible, massless string of constant length L. Part of this string rests on top of a cylinder of radius R and center at O. The string can slide on the cylinder without friction, and remains tense at all times. Furthermore, P and Q are also connected by a spring of zero length at rest and stiffness k. We define the inertial reference frame Oxy as shown in the figure. We denote A and B the points of tangency of the string with the cylinder; θ, ψ the polar angle of these points with respect to the Ox axis; and ξ the distance between A and P.



1. Write down the general expression of the position, velocity and acceleration vectors of P and Q relative to the Oxy reference frame as a function of θ, ψ, ξ and their time derivatives **(3 points)**.

   This is a planar motion problem of a system of two particles linked by one constraint (the string), so it has 3 effective degrees of freedom. We begin by defining some auxiliary unit vectors:

$$\begin{cases} \boldsymbol{e}_A = \cos\theta\boldsymbol{i} + \sin\theta\boldsymbol{j} \\ \boldsymbol{e}_\theta = -\sin\theta\boldsymbol{i} + \cos\theta\boldsymbol{j} \end{cases} \qquad \begin{cases} \boldsymbol{e}_B = \cos\psi\boldsymbol{i} + \sin\psi\boldsymbol{j}, \\ \boldsymbol{e}_\psi = -\sin\psi\boldsymbol{i} + \cos\psi\boldsymbol{j}, \end{cases} \tag{1}$$

   The position, velocity, and acceleration of P are:

$$\boldsymbol{r}_0^P = R\boldsymbol{e}_A - \xi\boldsymbol{e}_\theta \tag{2}$$
$$\boldsymbol{v}_0^P = \xi\dot{\theta}\boldsymbol{e}_A + (R\dot{\theta} - \dot{\xi})\boldsymbol{e}_\theta \tag{3}$$
$$\boldsymbol{a}_0^P = (2\dot{\xi}\dot{\theta} + \xi\ddot{\theta} - R\dot{\theta}^2)\boldsymbol{e}_A + (R\ddot{\theta} - \ddot{\xi} + \xi\dot{\theta}^2)\boldsymbol{e}_\theta \tag{4}$$

   The constraint set by the string means that the distance between B and Q is $L - \xi - R(\psi - \theta)$. Hence, for Q:

$$\boldsymbol{r}_0^Q = R\boldsymbol{e}_B + [L - \xi - R(\psi - \theta)]\boldsymbol{e}_\psi \tag{5}$$
$$\boldsymbol{v}_0^Q = [\xi\dot{\psi} + R\dot{\psi}(\psi - \theta) - L\dot{\psi}]\boldsymbol{e}_B + (R\dot{\theta} - \dot{\xi})\boldsymbol{e}_\psi \tag{6}$$
$$\boldsymbol{a}_0^Q = [2\dot{\xi}\dot{\psi} + R(\dot{\psi}^2 - 2\dot{\theta}\dot{\psi}) + \xi\ddot{\psi} + R\ddot{\psi}(\psi - \theta) - L\ddot{\psi}]\boldsymbol{e}_B + [\xi\dot{\psi}^2 + R\dot{\psi}^2(\psi - \theta) - L\dot{\psi}^2 + R\ddot{\theta} - \ddot{\xi}]\boldsymbol{e}_\psi \tag{7}$$

2. Write down the expression of all forces acting on P and Q **(3 points)**.

   The forces acting on each particle are weight, spring force, and string tension. The magnitude of the

tension on P and Q is identical, since the string has no friction over the cylinder:

$$\boldsymbol{W}_P = \boldsymbol{W}_Q = -mg\boldsymbol{j} \tag{8}$$

$$\boldsymbol{F}_{S,P} = -\boldsymbol{F}_{S,Q} = k\boldsymbol{PQ} = k(\boldsymbol{r}_0^Q - \boldsymbol{r}_0^P) = kR(\boldsymbol{e}_B - \boldsymbol{e}_A) + k[L - \xi - R(\psi - \theta)]\boldsymbol{e}_\psi + k\xi\boldsymbol{e}_\theta \tag{9}$$

$$\boldsymbol{T}_P = T\boldsymbol{e}_\theta \tag{10}$$

$$\boldsymbol{T}_Q = -T\boldsymbol{e}_\psi \tag{11}$$

3. Discuss whether the mechanical energy of P is conserved; if so, write down the expression of the corresponding potential energy **(1 point)**.

   The mechanical energy of the individual particles is not conserved, because the tension force, which is not conservative, exerts work on them (the particles P and Q have a velocity component along $\boldsymbol{e}_\theta$ and $\boldsymbol{e}_\psi$, respectively).

   Extra credit: the mechanical energy *of the system* $P + Q$ is conserved. The tension force is an internal force to the system $P + Q$ (and the cylinder, which experiences no change of mechanical energy), and the work it exerts on one particle is equal and opposite to the work it exerts on the other. Indeed, it is easy to check that $\boldsymbol{T}_P \cdot \boldsymbol{v}_0^P + \boldsymbol{T}_Q \cdot \boldsymbol{v}_0^Q = 0$. The potential energy for the whole system can be written as:

$$V = m(\boldsymbol{r}_0^P + \boldsymbol{r}_0^Q) \cdot \boldsymbol{g} + \frac{1}{2}k|\boldsymbol{PQ}|^2 \tag{12}$$

4. Consider now that we restrict the motion of the system to symmetric configurations about the Oy axis only. What is the number of effective degrees of freedom now? Find the mathematical relations that exist between θ, ψ, ξ in this case **(1.5 points)**.

   When the configuration is symmetric about the Oy axis, we only need to study one of the particles to analyze the motion of the system. There is only one effective degree of freedom, since we can express ψ and ξ as functions of θ:

$$\psi = \pi - \theta; \quad \xi = \theta R + \frac{L - \pi R}{2}. \tag{13}$$

5. Find the approximate value of θ, ψ, ξ when the system is in equilibrium and the configuration is symmetric about the Oy axis. To do so, assume that $0 < |\theta| \ll 1$ and linearize the equations of motion **(1.5 points)**.

   We need to project the equation of Newton's second law for one of the particles, e.g. P, along two independent directions, e.g. $\boldsymbol{i}$ and $\boldsymbol{j}$, and make all time derivatives equal to zero:

$$\boldsymbol{i}: \quad -T\sin\theta - k[2R\cos\theta + (2\theta R + L - \pi R)\sin\theta] = 0 \tag{14}$$

$$\boldsymbol{j}: \quad T\cos\theta - mg = 0 \tag{15}$$

   Linearizing the equations about $\theta = 0$ yields:

$$\boldsymbol{i}: \quad -T\theta - k[2R + (L - \pi R)\theta] \simeq 0 \tag{16}$$

$$\boldsymbol{j}: \quad T - mg \simeq 0 \tag{17}$$

   Hence:

$$T = mg \tag{18}$$

$$\theta = -\frac{2}{mg/(kR) + L/R - \pi} \tag{19}$$

   It is easy to see that in order for $\theta \ll 1$ to hold, it is required that $mg/(kR) + L/R \gg \pi$. This happens when the spring characteristic force is sufficiently low compared to gravity, or when the string length is sufficiently large compared to the cylinder radius.