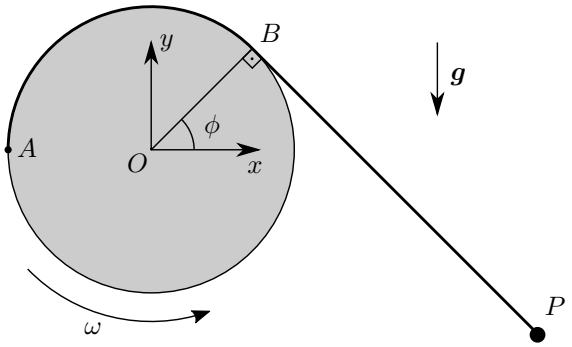
A heavy particle P of mass m is connected by a string of total length $l = 4\pi a$ to a cylindrical spool of radius a and center at O as shown in the figure. The spool rotates at a known constant angular rate ω counterclockwise. The string is massless, inelastic, thin, and remains always in tension during the motion of the particle. We define the inertial reference frame Oxy as shown in the figure. The other end of the string is attached to the spool at a point A of its perimeter, and we shall call B the instantaneous point of tangency between the straight segment of the string (the segment BP) and the spool. We will denote the angle between the local horizontal (axis Ox) and the radius OB as ϕ. Initially, $\boldsymbol{OA} = -a\boldsymbol{i}$, and B never reaches A.



1. Discuss the number of degrees of freedom of the motion of the particle **(1 point)**.

   This is a planar motion problem and the particle is constrained by the string, which removes one degree of freedom. The particle motion has therefore only one degree of freedom.

2. Write down the position, velocity and acceleration vectors of P with respect to Oxy as a function of the constants of the problem, time, the angle ϕ, and its derivatives **(3 points)**.

   It is easy to construct the position of P if we know the polar angle of A ($\theta_A = \pi + \omega t$) and B ($\theta_B \equiv \phi$). The fraction of string rolled up on the spool between A and B is equal to

   $$a(\pi + \omega t - \phi), \tag{1}$$

   hence, the rest of the string is the distance from B to P. We define the rotated vector basis that follows the motion of point B (primed vectors):

   $$\begin{cases} \boldsymbol{i}' = \cos\phi\boldsymbol{i} + \sin\phi\boldsymbol{j}, \\ \boldsymbol{j}' = -\sin\phi\boldsymbol{i} + \cos\phi\boldsymbol{j}. \end{cases} \tag{2}$$

   Then, the position, velocity and acceleration with respect to Oxy (taking derivatives in Oxy and applying Coriolis theorem where necessary):

   $$\boldsymbol{r}_P = \boldsymbol{OB} + \boldsymbol{BP} = a\boldsymbol{i}' - a(3\pi - \omega t + \phi)\boldsymbol{j}', \tag{3}$$

   $$\boldsymbol{v}_P = a(3\pi - \omega t + \phi)\dot{\phi}\boldsymbol{i}' + a\omega\boldsymbol{j}', \tag{4}$$

   $$\boldsymbol{a}_P = a[(3\pi - \omega t + \phi)\ddot{\phi} - 2\omega\dot{\phi} + \dot{\phi}^2]\boldsymbol{i}' + a(3\pi - \omega t + \phi)\dot{\phi}^2\boldsymbol{j}'. \tag{5}$$

3. Write down the mathematical expression of the forces acting on P **(2 points)**.

   There are two forces acting on P: its weight and the tension of the string. Since the magnitude of the tension is yet unknown, we introduce it as a parameter:

   $$\boldsymbol{W} = -mg\boldsymbol{j} = -mg\sin\phi\boldsymbol{i}' - mg\cos\phi\boldsymbol{j}', \tag{6}$$

   $$\boldsymbol{T} = T\boldsymbol{j}'. \tag{7}$$

4. Is the mechanical energy of P conserved? If so, write down the equation of conservation of mechanical energy; otherwise, explain why it is not conserved **(1 point)**.

   It is not conserved unless $\omega = 0$, because $\boldsymbol{T}$ does work on the particle and it is a non-conservative force.

5. Write down a closed set of equations that, if solved, would yield ϕ (and the other unknowns of the problem, if any) as a function of time. Note that you do not need to solve the equations **(2 points)**.

   The second law of Newton projected along $\boldsymbol{i}'$ and $\boldsymbol{j}'$ provides us with two independent second-order ordinary differential equations:

$$\boldsymbol{i}': \quad a[(3\pi - \omega t + \phi)\ddot{\phi} - 2\omega\dot{\phi} + \dot{\phi}^2] = -g\sin\phi, \tag{8}$$

$$\boldsymbol{j}': \quad ma(3\pi - \omega t + \phi)\dot{\phi}^2 = -mg\cos\phi + T. \tag{9}$$

   This choice of projection directions is interesting, as the first equation can be integrated to obtain $\phi(t)$ independently of T; the second equation would then yield T once $\phi(t)$ is known.

6. Consider now that $\omega = 0$. Can you write down a first integral of motion that only involves ϕ and $\dot{\phi}$ (but not $\ddot{\phi}$) in this case? You may want to rethink your previous answers for this particular case **(1 point)**.

   If $\omega = 0$, the mechanical energy of P is *conserved. We may write:*

$$\frac{1}{2}m[a(3\pi + \phi)]^2\dot{\phi}^2 + mg[a\sin\phi - a(3\pi + \phi)\cos\phi] = E \tag{10}$$

   Incidentally, it can be checked that by taking derivatives in this expression one obtains Eq. (8)*.*