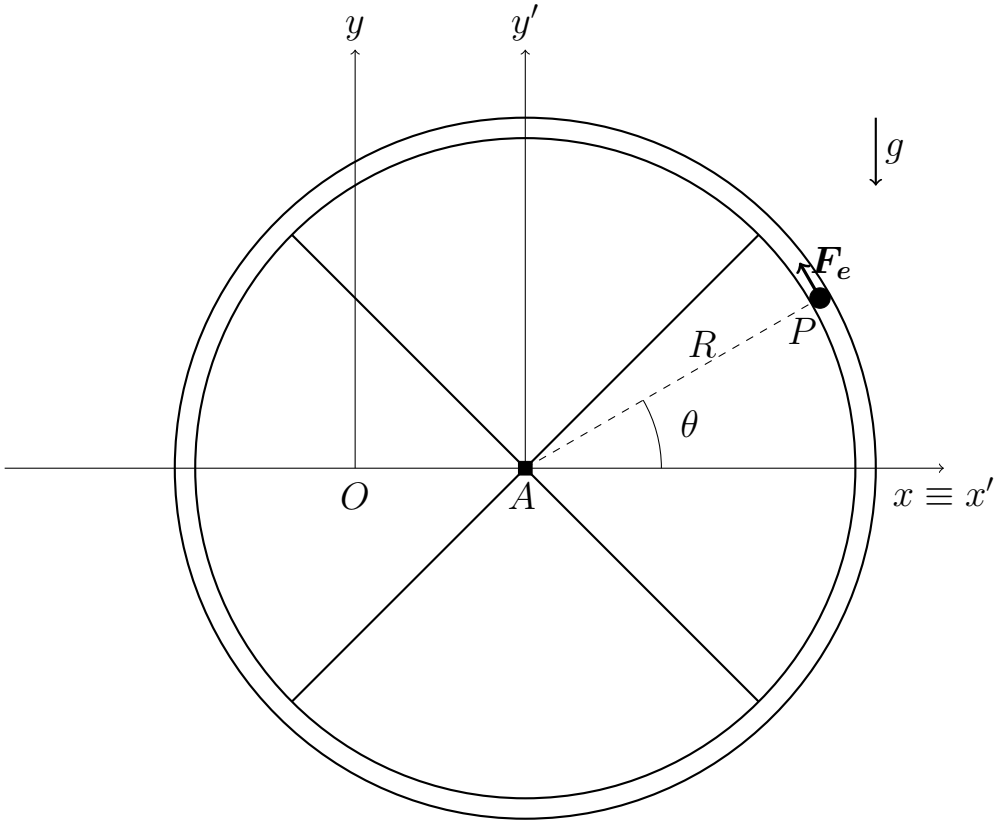
A heavy vehicle P of mass m moves along a smooth circular track of radius R. The track is contained on a vertical plane and it translates horizontally without rotation with respect to a fixed reference frame Oxy ($\boldsymbol{i}, \boldsymbol{j}$ are the unit vectors along the x and y directions, respectively). The position of the centre of the track, A, with respect to Oxy is given by $\boldsymbol{r_A} = h(t)\boldsymbol{i}$, where $h(t)$ is a known function of time. A moving reference frame $Ax'y'$ is defined as indicated in the figure. We model the vehicle as a point particle. The vehicle moves with respect to the track using its engine that provides a force $\boldsymbol{F_e}(t)$, which is a function of time. The force direction is always parallel to the tangent direction to the circular track.

1. Determine the relative velocity and relative acceleration of P as a function of θ and its time derivatives. (1 point)
2. Determine the fictitious forces of inercia acting on P. (1 point)
3. Assuming that $\boldsymbol{F_e}(t)$ is known, write down a system of 2 coupled ordinary differential equations that would provide the position of the particle and the reaction from the track, both as a function of time. Note that you do not have to solve this system. (3 points)

Consider now that $h(t) = R(1 - \cos(\omega t))$, where ω is a positive constant. At $t = 0$, the relative position of the vehicle is $\boldsymbol{r'_P} = R\boldsymbol{j}$ and the vehicle is moving counterclockwise.

4. Determine the force that the engine needs to provide as a function of time, $\boldsymbol{F_e}(t)$, in order for the vehicle to move at a constant relative speed, ωR. (2 points)
5. Compute the reaction from the track as a function of time. (1 point)
6. Compute the absolute velocity of the vehicle with respect to Oxy as a function of time. (1 point)
7. If the engine of the vehicle is only able to push the vehicle forward, is it possible to move at a constant relative speed along the track? (1 point)



1. First, we define the unit vectors in polar coordinates

$$\boldsymbol{e_r} = \cos\theta \boldsymbol{i} + \sin\theta \boldsymbol{j}, \tag{1}$$
$$\boldsymbol{e_\theta} = -\sin\theta \boldsymbol{i} + \cos\theta \boldsymbol{j}. \tag{2}$$

Then the relative position, velocity and acceleration of the vehicle are given by

$$\boldsymbol{r'_P} = R\boldsymbol{e_r}, \tag{3}$$
$$\boldsymbol{v'_P} = R\dot{\theta}\boldsymbol{e_\theta}, \tag{4}$$
$$\boldsymbol{a'_P} = -R\dot{\theta}^2\boldsymbol{e_r} + R\ddot{\theta}\boldsymbol{e_\theta}. \tag{5}$$

2. Since the moving reference frame is always parallel to the fixed reference frame, the angular velocity is 0. Therefore, the only non-zero component of the fictitious force is

$$\boldsymbol{F_I} = -m\boldsymbol{a_A} = -m\frac{\mathrm{d}^2h(t)}{\mathrm{d}t^2}\boldsymbol{i}. \tag{6}$$

3. The equations required are given by Newton's second law applied to a moving reference frame $\sum \boldsymbol{F} + \sum \boldsymbol{F_I} = m\boldsymbol{a'_P}$. We are going to use the transformations

$$\boldsymbol{i} = \cos\theta \boldsymbol{e_r} - \sin\theta \boldsymbol{e_\theta}, \tag{7}$$
$$\boldsymbol{j} = \sin\theta \boldsymbol{e_r} + \cos\theta \boldsymbol{e_\theta}. \tag{8}$$

The fictitious force and the relative acceleration we have already computed, but we need to express them in a common system, so that we transform the fictitious force to polar coordinates

$$\boldsymbol{F_I} = -m\frac{\mathrm{d}^2h(t)}{\mathrm{d}t^2}(\cos\theta \boldsymbol{e_r} - \sin\theta \boldsymbol{e_\theta}). \tag{9}$$

The external forces are the weight, the engine force and the normal reaction. The three forces can be written

$$\boldsymbol{F_w} = -mg\boldsymbol{j} = -mg(\sin\theta \boldsymbol{e_r} + \cos\theta \boldsymbol{e_\theta}), \tag{10}$$
$$\boldsymbol{F_e} = F_e(t)\boldsymbol{e_\theta}, \tag{11}$$
$$\boldsymbol{N} = N\boldsymbol{e_r}. \tag{12}$$

So that the 2 equations are

$$N - mg\sin\theta - m\frac{\mathrm{d}^2h(t)}{\mathrm{d}t^2}\cos\theta = -mR\dot{\theta}^2, \tag{13}$$
$$F_e(t) - mg\cos\theta + m\frac{\mathrm{d}^2h(t)}{\mathrm{d}t^2}\sin\theta = mR\ddot{\theta}. \tag{14}$$

4. If the vehicle moves at constant relative speed, then, from eq. (4), it is easy to see that $\dot{\theta} = \omega$. Note that this implies $\ddot{\theta} = 0$. Integrating, and imposing the initial condition, we obtain

$$\theta(t) = \omega t + \frac{\pi}{2}. \tag{15}$$

Recall that

$$\sin\theta = \sin\left(\omega t + \frac{\pi}{2}\right) = \cos(\omega t), \tag{16}$$
$$\cos\theta = \cos\left(\omega t + \frac{\pi}{2}\right) = -\sin(\omega t). \tag{17}$$

Now, we are given $h(t)$ so that

$$\frac{\mathrm{d}^2h(t)}{\mathrm{d}t^2} = R\omega^2\cos(\omega t). \tag{18}$$

From eq. (14), we obtain

$$F_e(t) = mg\cos\theta - mR\omega^2\cos(\omega t)\sin\theta = -mg\sin(\omega t) - mR\omega^2\cos^2(\omega t). \tag{19}$$

5. From eq. (13), we obtain

$$N = mg\sin\theta + mR\omega^2\cos(\omega t)\cos\theta - mR\omega^2 = mg\cos(\omega t) - mR\omega^2\left(1 + \frac{1}{2}\sin(2\omega t)\right). \tag{20}$$

6. The absolute velocity of the vehicle is given by $\boldsymbol{v_P} = \boldsymbol{v_A} + \boldsymbol{v'_P}$. We can compute both terms and obtain

$$\boldsymbol{v_P} = R\omega\sin(\omega t)\boldsymbol{i} + R\omega\boldsymbol{e_\theta} = R\omega[(\sin(\omega t) - \cos(\omega t))\boldsymbol{i} - \sin(\omega t)\boldsymbol{j}]. \tag{21}$$

7. Analyzing eq. (19) it is easy to see that already at $t = 0$ the required force to keep a constant relative force is negative, i.e. pushing the vehicle backwards. Therefore if the vehicle can only push forward it is not possible to move at constant relative speed.