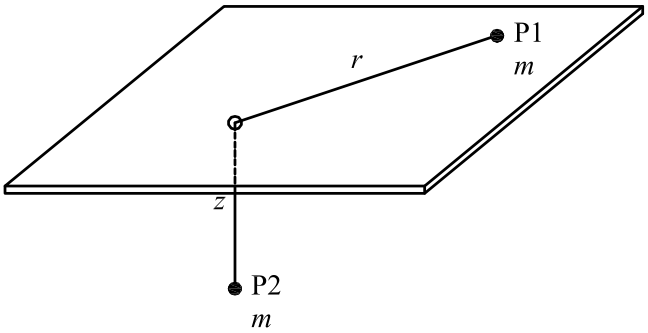
1. We consider a smooth horizontal plane wih a small hole in it. One of the ends of a massless cord of length $2a$ is introduced in the hole and then, two point particles, P1 and P2, of the same mass m are attached to the ends of the cord. Therefore, P1 is constrained to move on the plane and P2 moves in the vertical line below the hole. Initially, half of the cord lies under the plane and half of the cord is tight on the plane. At that instant P1 is pushed with a velocity v_0 perpendicular to the cord.

   - What should be the value of v_0 if P1 describes a circumference? (1 point)
   - In that case, what is the value of the tension of the cord? (1 point)
   - If $v_0 = \sqrt{8ag/3}$, reduce the problem to quadratures. (Hint: you might use the energy equation applied to the system of two particles, and recall that in such a case internal forces do not perform work.) (3 points)
   - Show that in that case P1 oscillates between $r = a$ and $r = 2a$. (2 points)
   - Apply Newton's law to P2 and show that the tension of the cord is $T = mg(\frac{1}{2} + \frac{4a^3}{3r^3})$ (1 point)
   - Write down the new equations of motion if the plane on which P1 moves is rough, with a friction coefficient μ. (2 points)

   (Hint: the roots of the polynomial $-3x^3 + 7x^2 - 4$ are $x_1 = 1$, $x_2 = 2$ and $x_3 = -2/3$)



The equations of motion using r, z, θ and their derivatives as the state variables are:

$$m\left(\ddot{r} - r\dot{\theta}^2\right) = -T \quad (1)$$

$$m\left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right) = 0 \quad (2)$$

$$m\ddot{z} = mg - T \quad (3)$$

Tension is associated to the constraint $z + r = 2a$. From (2) the conservation of the angular momentum can be deduced. Thus: $r^2\dot{\theta} = C$ (C constant)

(a) If r is constant, then $\dot{r} = \ddot{r} = 0$. In addition, differenciating $r + z = 2a$, we get:

$$\ddot{z} = -\ddot{r} \quad (4)$$

Therefore, $\ddot{z} = 0$. Using this value in (3), we obtain $T = mg$. Susbstituting T, $\ddot{r}$ and imposing the initial conditions in (1) we finally end up with:

$$-mr\dot{\theta}^2 = -mg \Rightarrow \frac{mv_0}{a} = mg \Rightarrow v_0 = \sqrt{ag} \quad (5)$$

(b) The tension on the cord is, as seen before, $T = mg$

(c) In order to reduce to quadratures, two different paths can be followed. First option, using the equations of motion, we can subtract (1) - (3) and use the fact that $r^2\theta = C = av_0$, to get:

$$\ddot{r} - \frac{a^2v_0^2}{2r^3} + \frac{g}{2} = 0 \quad (6)$$

Multiplying by $2\dot{r}$ and integrating we arrive to:

$$\dot{r}^2 + \frac{a^2v_0^2}{2r^2} + gr + D = 0 \quad (7)$$

where $D = -\frac{v_0^2}{2} - ga$ (imposing the initial conditions). Using the data $v_0^2 = 8ag/(3)$, the previous equation reads:

$$\dot{r}^2 + \frac{4}{3}\frac{a^3g}{r^2} + gr - \frac{7}{3}ag = 0 \quad (8)$$

Separating the variables r and t we finally get:

$$\int_a^r \frac{rdr}{\sqrt{P(r/a)}} = \sqrt{\frac{a^3g}{3}}t \quad (9)$$

where

$$P(r/a) = -4 - 3\left(\frac{r}{a}\right)^3 + 7\left(\frac{r}{a}\right)^2$$

Second option, we can make use of the fact the energy does not vary. This is due to the fact that the external force (gravity) derive from a potential and the reactions (tension and normal to the plane) do not work (when the whole system is analyzed) Then, we have:

$$\frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2) - mgz = \frac{1}{2}mv_0^2 - mga$$

Using $\dot{z} = -\dot{r}$, $z = 2a - r$ and $r^2\dot{\theta} = av_0$, we arrive to (7):

$$\dot{r}^2 + \frac{a^2v_0^2}{2r^2} + gr + D = 0 \quad (10)$$

From this point on, the derivation is the same as before.

(d) The variable $\dot{r}^2$ should be positive at any instant. This is equivalent to state that $P(r/a) \geq 0$. The statement of the problem indicates us that the positive roots of this polynomial are $r/a = 1$ and $r/a = 2$. For values between those roots $P(r/a)$ is positive (e.g., $r/a = 3/2 \rightarrow P = 13/8$). Therefore, the motion can only take place between $r = a$ and $r = 2a$.

(e) From (3), using (4) and (6) we have:

$$T = \frac{m}{2}\left(g + \frac{a^2 v_0^2}{r^3}\right) \tag{11}$$

And with $v_0^2 = 8ag/3$, the final result is:

$$T = mg\left(\frac{1}{2} + \frac{4}{3}\left(\frac{a}{r}\right)^3\right)$$

(f) When the plane on which P_1 is moving is not smooth but rough, a new term accounting for the resistance should be included. This term has the following form:

$$\boldsymbol{R} = -\mu N \frac{\boldsymbol{v}}{v} \tag{12}$$

And $N = mg$ (from the equation of P_1 in vertical direction). Then, the equations of motion (1-3) are rewritten as:

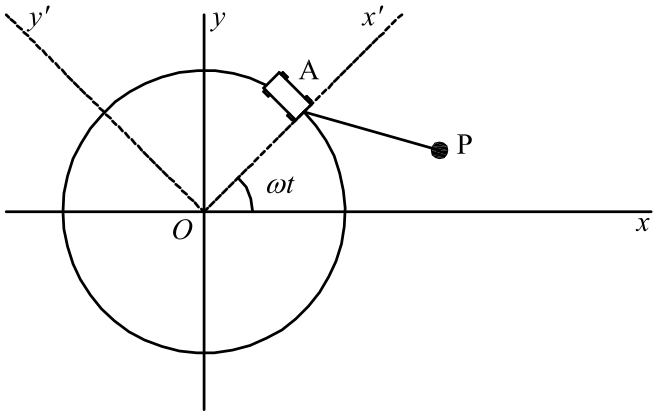
$$m\left(\ddot{r} - r\dot{\theta}^2\right) = -T - \mu mg \frac{\dot{r}}{\sqrt{\dot{r}^2 + r^2\dot{\theta}^2}} \tag{13}$$

$$m\left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right) = -\mu mg \frac{r\dot{\theta}}{\sqrt{\dot{r}^2 + r^2\dot{\theta}^2}} \tag{14}$$

$$m\ddot{z} = mg - T \tag{15}$$

2. On an perfectly smooth horizontal plane, a skater P is towed by a vehicle A. The vehicle is moving along a circumference whose center is in O and its radius is a, with constant angular velocity ω. The skater is attached to the vehicle by a bar without mass and longitude a. Initially, the vehicle is located at $(a, 0)$ and the skater is at rest in $(2a, 0)$. Defining a rotating reference frame $Ox'y'$ with origin in O and axes OA and the line perpendicular to OA, determine:

   - The relative acceleration of the skater with respect to $Ox'y'$ as a function of ψ and its time derivatives. ψ is the angle between the bar AP and the axis Ox' (1 point)
   - The centrifugal and Coriolis forces also as a function of ψ and its time derivatives. (2 points)
   - Determine the initial condition for ψ and $\dot{\psi}$. (1 point)
   - Reduce the evolution of ψ to quadratures. (2 points) (Hint: consider that $\dot{\psi}^2 = \int 2\dot{\psi}\ddot{\psi}dt$)
   - Determine the tension on the bar. (2 points)
   - If the rigid bar is substituted by a string which cannot bear compression. Determine if the string is ever subject to compression. If yes specify precisely when it happens. (2 points)



(a) There are two options to express the relative position of the skater. The first uses only $Ox'y'$:

$$\boldsymbol{r} = a(1+\cos\psi)\boldsymbol{i'} - a\sin\psi\boldsymbol{j'} \tag{16}$$

The second expression is a bit *ad hoc* and makes use of polar coordinates to express $\boldsymbol{AP}$

$$\boldsymbol{r} = a\boldsymbol{i'} + a\boldsymbol{u}_r \tag{17}$$

Using the second facilitates the computation but the results are, of course, equivalent. For the first:

$$\begin{aligned}\dot{\boldsymbol{r}} &= -a\sin\psi\dot{\psi}\boldsymbol{i'} - a\cos\psi\dot{\psi}\boldsymbol{j'}\\ \ddot{\boldsymbol{r}} &= -\left(a\cos\psi\dot{\psi}^2 + a\sin\psi\ddot{\psi}\right)\boldsymbol{i'} + \left(a\sin\psi\dot{\psi}^2 - a\cos\psi\ddot{\psi}\right)\boldsymbol{j'}\end{aligned}$$

For the second:

$$\begin{aligned}\dot{\boldsymbol{r}} &= a\dot{\psi}\boldsymbol{u}_\psi\\ \ddot{\boldsymbol{r}} &= -a\dot{\psi}^2\boldsymbol{u}_r + a\ddot{\psi}\boldsymbol{u}_\psi\end{aligned}$$

(b) The vector expression for the centrifugal force is: $\boldsymbol{F}_{cf} = -m\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r})$. Substituting $\boldsymbol{\omega} = \omega\boldsymbol{k}$ and $\boldsymbol{r}$, we obtain either:

$$\boldsymbol{F}_{cf} = ma\omega^2\left[(1+\cos\psi)\boldsymbol{i'} - \sin\psi\boldsymbol{j'}\right] \tag{18}$$

or:

$$\boldsymbol{F}_{cf} = ma\omega^2\left[\boldsymbol{i'} + \boldsymbol{u}_r\right] \tag{19}$$

In turn, the Coriolis force is: $\boldsymbol{F}_C = -2\boldsymbol{\omega}\times\dot{\boldsymbol{r}}$. Substituting, we get either:

$$\boldsymbol{F}_C = -2ma\omega\left(-\sin\psi\dot{\psi}\boldsymbol{j'} + \cos\psi\dot{\psi}\boldsymbol{i'}\right) \tag{20}$$

or:

$$\boldsymbol{F}_C = -2ma\omega\dot{\psi}\boldsymbol{u}_r \tag{21}$$

(c) From the initial position of skater and vehicle $\psi_0 = 0$. Initially, the absolute velocity of the skater is zero but the axes $Ox'y'$ are rotating at a constant rate ω. Using the expression of relative motion for the velocity we arrive at the following identity

$$-a\dot{\psi}_0 + 2a\omega = 0 \Rightarrow \dot{\psi}_0 = 2\omega \tag{22}$$

(d) The equations of motion should be posed to analyze the motion of the skater. In addition to the fictitious forces, the tension must be considered. Apart from that, no external forces are involved. Again, we can use the first of the considered reference systems to express the equations of motion:

$$-ma\left(\cos\psi\dot{\psi}^2 + \sin\psi\ddot{\psi}\right) = -T\cos\psi + ma\omega^2(1+\cos\psi) - 2ma\omega\cos\psi\dot{\psi} \tag{23}$$

$$ma\left(\sin\psi\dot{\psi}^2 - \cos\psi\ddot{\psi}\right) = T\sin\psi - ma\omega^2\sin\psi + 2ma\omega\sin\psi\dot{\psi} \tag{24}$$

or:

$$-ma\dot{\psi}^2 = -T + ma\omega^2\cos\psi + ma\omega^2 - 2ma\omega^2\dot{\psi} \tag{25}$$

$$ma\ddot{\psi} = -ma\omega^2\sin\psi \tag{26}$$

The second formulation of the equations of motion is more convenient because the tension (which is unknown) is only present in one equation. From (23-24) we can obtain (25-26) by operating them in the following way: (25) = (23) $\cos\psi$ - (24) $\sin\psi$ and (26) = - (23) $\sin\psi$ - (24) $\cos\psi$.

In (26), T is not present. Multiplying by $2\dot{\psi}dt$ both sides of the equation, we can use the relationship of the statement:

$$2\dot{\psi}\ddot{\psi}dt = -2\dot{\psi}\omega^2 \sin\psi dt \Rightarrow \dot{\psi}^2 = 2\omega^2 \cos\psi + C \tag{27}$$

Substituting $\psi_0 = 0, \dot{\psi}_0 = 2\omega$ in (27), we obtain:

$$\dot{\psi}^2 = 2\omega^2(1 + \cos\psi) \tag{28}$$

And finally, the expression of the quadrature for ψ is:

$$\frac{d(\psi/2)}{\cos(\psi/2)} = \omega dt \tag{29}$$

(e) Using (25), and solving for T, we get:

$$T = ma\left(\dot{\psi}^2 + \omega^2(1 + \cos\psi) - 2\omega\dot{\psi}\right) \tag{30}$$

We can substitute $\dot{\psi}$ from (28) in the previous expression to obtain:

$$T = ma\omega^2 \cos\frac{\psi}{2}\left(3\cos\frac{\psi}{2} - 2\right) \tag{31}$$

where the identity $1 + \cos\psi = 2\cos^2(\psi/2)$ has been used.

(f) From (31), we can deduce that:

$$T = 0 \Rightarrow \begin{cases} \cos(\psi/2) = 0 \rightarrow \psi = \pi \\ \cos(\psi/2) = \frac{2}{3} \rightarrow \psi = 2\arccos(\frac{2}{3}) \end{cases} \tag{32}$$