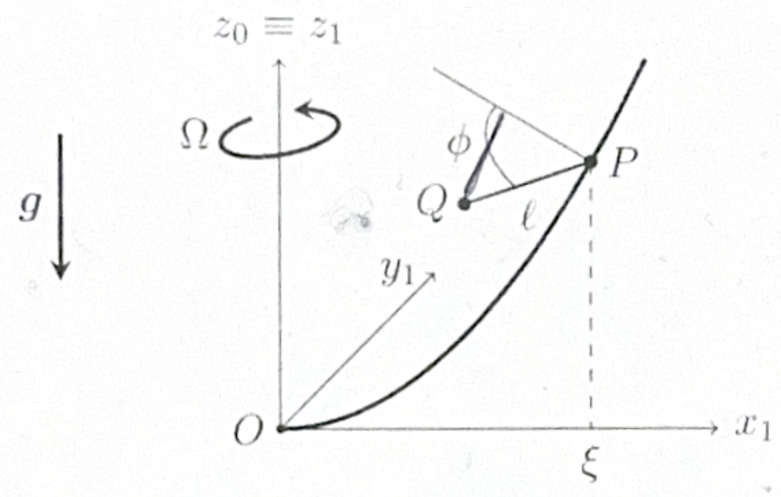
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PROBLEM 1

10 points, 60 minutes.

Two heavy point particles P and Q, each of mass m, are joined by a massless, rigid bar of length ℓ. Particle P slides without friction over a wire, which has the shape of the parabola $z_1 = x_1^2$, shown in the figure. The wire is fixed in a frame S_1 that rotates about the vertical axis at a known, constant angular velocity Ω with respect to an inertial reference frame S_0 (not shown). The bar that joins P and Q remains perpendicular to the wire at all times. We introduce the angle ϕ as the angle between the bar and the Ox_1z_1 plane. We shall use the generalized coordinates $\xi \equiv x_1^P$ and ϕ to describe the configuration of the system.



1. Find the expression of the unit tangent, normal, and binormal vectors $\boldsymbol{e}_t$, $\boldsymbol{e}_n$, $\boldsymbol{e}_b$ to the parabola, as a function of ξ. Find also the curvature κ as a function of ξ (**2 points**).
2. Write down the general expressions of the position and velocity vectors of P and Q with respect to S_1, as a function of ξ, ϕ and their derivatives (**2 points**).
3. Write down all forces acting on P and Q for an observer in S_1, introducing new unknowns as needed (**4 points**).
4. Is the mechanical energy of the system conserved in S_1? Discuss. If it is conserved, write down the expression of the mechanical energy (**2 points**).

Hint: You may use the symbol $b = \sqrt{4\xi^2 + 1}$ to keep your expressions more compact if you like.

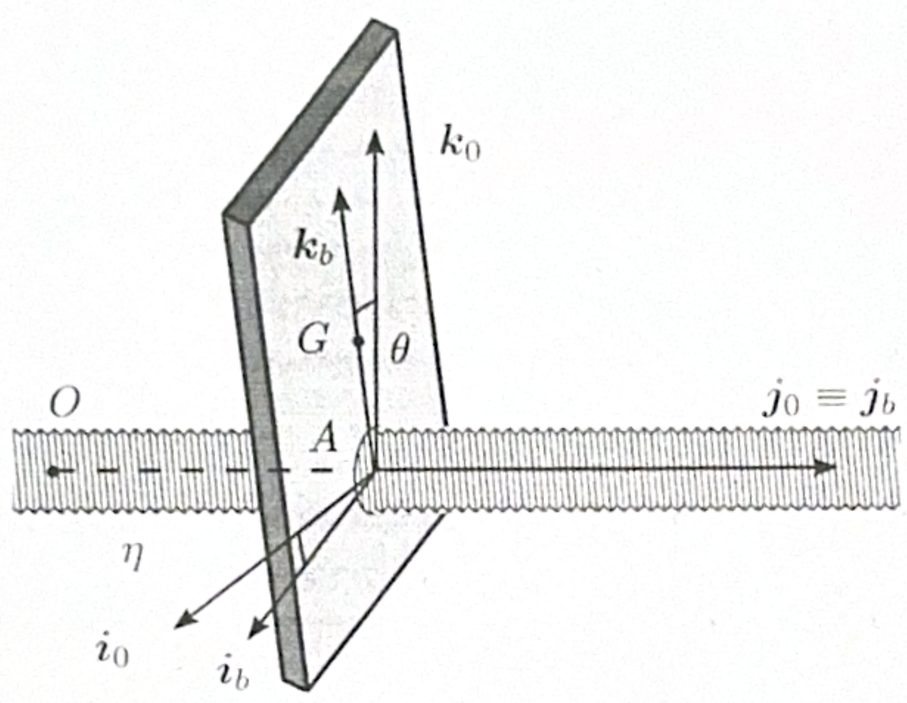
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PROBLEM 2

10 points, 60 minutes.

A heavy rectangular plate b has a threaded hole centered on point A. We introduce the body-fixed reference frame $S_b\{A; \boldsymbol{i}_b, \boldsymbol{j}_b, \boldsymbol{k}_b\}$ as shown in the figure. The center of mass G of b is located at a distance ℓ of A along the $\boldsymbol{k}_b$ direction. The mass of b is m, and its moment of inertia about the axis of the threaded hole is I_A. The basis $\{\boldsymbol{i}_b, \boldsymbol{j}_b, \boldsymbol{k}_b\}$ is a principal basis of inertia at A. As shown below, the body b slides without friction over a bar with a matching thread, which is at rest in the inertial frame $S_0\{O; \boldsymbol{i}_0, \boldsymbol{j}_0, \boldsymbol{k}_0\}$. The thread pitch is h (i.e., for every full rotation about the bar, the body b is displaced a horizontal distance h). We shall model the reaction of the bar on b as an unknown force on A and an unknown a torque, $\boldsymbol{F}_A$ and $\boldsymbol{Q}_A$. To describe the configuration of the system, we introduce as parameters the horizontal position η of A, and the angle θ of the vector $\boldsymbol{AG}$ with respect to the vertical as shown.



1. Express vectors $\{\boldsymbol{i}_b, \boldsymbol{j}_b, \boldsymbol{k}_b\}$ in terms of $\{\boldsymbol{i}_0, \boldsymbol{j}_0, \boldsymbol{k}_0\}$ as a function of θ (**1 point**).
2. Express the mathematical constraint between η and θ. Assume that $\theta = 0$ when $\eta = 0$ (**1.5 points**).
3. Write down the position, velocity, and acceleration of G, and the angular velocity an acceleration of b, in S_0, as a function of θ and its derivatives (**1.5 points**).
4. Write down all the applied and reaction forces and torques on the body, indicating the point of application of the forces. Introduce new unknowns as needed. Taking into account the thread pitch h, express the reaction force along $\boldsymbol{j}_0$ and the reaction torque along $\boldsymbol{j}_0$ in terms of only one extra unknown. **Hint**: observe that this is a stationary constraint and therefore its reactions do not exert work (**2.5 points**).
5. Write down the equations of motion of b. Find one equation for $\ddot{\theta}$ that does not include any reaction constraints (**2.5 points**).
6. Linearize this equation of motion about $\theta = \pi$ and find the natural frequency of small oscillations about this stable equilibrium configuration (**1 point**).

$\vec{c}_b = \cos\theta\,\vec{c}_0 - \sin\theta\,\vec{k}_0 \rightarrow -\dot{\theta}\,\vec{k}_b$

$\vec{j}_b = \vec{j}_0$

$\vec{k}_b = \sin\theta\,\vec{c}_0 + \cos\theta\,\vec{k}_0 \rightarrow \dot{\theta}\,\vec{c}_b$