SHORT QUESTIONS

10 points, 75 minutes.

Please answer the questions below. Questions requiring a True/False answer also require a justification of your answer. **A right answer without explanation will not be graded**.

1. An airplane P describes a planar trajectory, with a tangential acceleration $a_t = 1$ and a normal acceleration $a_n = 1$. At time $t = 0$, it is located at the origin, its speed is $v_0^P(0) = 1$ and the tangent and normal unit vectors are $\boldsymbol{e}_t(0) = \boldsymbol{i}$ and $\boldsymbol{e}_n(0) = \boldsymbol{j}$ **(1.5 points)**.

   (a) Write down the expression of the speed, curvature, tangent vector, and normal vector of the airplane for all times $t \geq 0$.
   The speed is $v_0^P = 1 + t$. The curvature is $\kappa = 1/(1+t)$. Frenet's equations in this case read

   $$\frac{\mathrm{d}\boldsymbol{e}_t}{\mathrm{d}t} = \boldsymbol{e}_n; \quad \frac{\mathrm{d}\boldsymbol{e}_n}{\mathrm{d}t} = -\boldsymbol{e}_t$$

   from where

   $$\boldsymbol{e}_t = \cos t\boldsymbol{i}_0 + \sin t\boldsymbol{j}_0; \quad \boldsymbol{e}_n = -\sin t\boldsymbol{i}_0 + \cos t\boldsymbol{j}_0$$

   (b) Write down the velocity vector $\boldsymbol{v}_0^P(t)$ of the airplane and integrate it to obtain its trajectory $\boldsymbol{r}_0^P(t)$.
   The velocity and position vectors are (integrating by parts):

   $$\boldsymbol{v}_0^P = (1+t)\cos t\boldsymbol{i}_0 + (1+t)\sin t\boldsymbol{j}_0;$$
   $$\boldsymbol{r}_0^P = (t\sin t + \sin t + \cos t)\boldsymbol{i}_0 - (t\cos t - \sin t + \cos t)\boldsymbol{j}_0$$

2. A point particle P of mass m is constrained to move on the surface of a frictionless spherical planet of radius R that spins about its vertical axis with a constant angular velocity Ω. We define the planet-fixed reference frame S_1 that rotates with the planet. Initially the particle is at the equator and moves Northward with speed v_1^P relative to the planet **(1.5 points)**.

   (a) Write down the equations of motion of the particle in S_1 using spherical coordinates.
   Taking into account the Coriolis and the centrifugal forces in the rotating reference frame, the equations are (normal forces take care of keeping $r = R$ constant)

   $$\ddot{\theta}\cos\phi - 2(\dot{\theta}+\Omega)\dot{\phi}\sin\phi = 0$$
   $$\ddot{\phi} + (\dot{\theta}+\Omega)^2\cos\phi\sin\phi = 0$$

   (b) Can the particle ever reach the North pole in its subsequent motion?
   No. A simple way to prove this is to consider the motion of the particle from an inertial reference frame S_0. The angular momentum of the particle about the center of the planet in S_0 is conserved, and given the rotation of the planet, its component along the vertical axis is initially non-zero. Hence, the particle can never reach the vertical axis, where the North pole is.

3. A particle P moves in the one-dimensional potential given by $V(x) = \sin(\pi/(1+x^2)) - 1/2$. Indicate which of the following statements are true or false and provide a brief justification **(2 points)**.

   (a) If P is released at $x = 1$ with zero velocity, the particle will begin to move.
   FALSE. $x = 1$ is a critical point of the potential, and thus it is an equilibrium point (albeit unstable).

(b) If P is released between $x = -1$ and $x = 1$, it will remain trapped by the potential.
FALSE. If the total mechanical energy of the particle is higher than $E^* = 1/2$, it will escape

(c) If for whatever reason P escapes to $x \to \infty$, it will accelerate to infinite speed.
FALSE. The potential is unbounded.

(d) If P is released with zero velocity at a point x^* where $V(x^*) = 0$, then the particle will remain at rest.
FALSE. The value of the potential is irrelevant to determine equilibrium positions. It is the gradient of the potential what matters.

4. A heavy elevator of mass m is hanging from a pulley with a massless elastic cable, whose stiffness is inversely proportional to its natural length: $k = k'/x_0$. The damping on the elevator motion is negligible. A boy of negligible mass is jumping up and down in an elevator, creating a periodic force $\boldsymbol{F}_1 = F_1 \cos(\omega_1 t)\boldsymbol{i}$ on it, with $F_1/(mg) < 1$ **(1.5 points)**.

(a) Write down the expression of $x(t)$ for long times, after any transient oscillations have died out.
The equation of motion for the elevator is

$$m\ddot{x} + k(x - x_0) = mg + F_1 \cos(\omega_1 t)$$

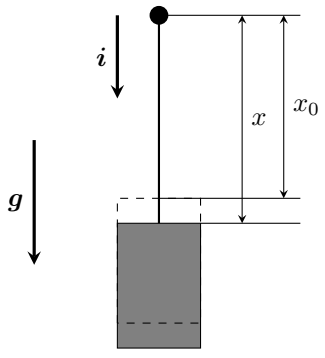
Integrating for the forced response:

$$x(t) = x_0 + \frac{mg}{k} + \frac{F_1/m}{k/m - \omega_1^2} \cos(\omega_1 t + \phi_1)$$

where $\phi_1 = 0$ if $k/m - \omega_1^2 > 0$ and $\phi_1 = \pi$ if $k/m - \omega_1^2 < 0$.

(b) Find the range of values of x_0 for which the cable does not lose its tension during the oscillations, as a function of the parameters of the problem
This happens when $x - x_0 = 0$ at some point in the cycle of the oscillations, i.e., when x_0 is within the limit values

$$\frac{mg}{k} = \frac{F_1/m}{|k/m - \omega_1^2|} \Rightarrow x_0 = \frac{k'}{m\omega_1^2}\left(1 \pm \frac{F_1}{mg}\right)$$



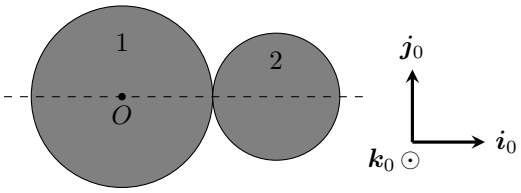
5. The body-fixed reference frame of a satellite has a vector basis $B_b = \{\boldsymbol{i}_b, \boldsymbol{j}_b, \boldsymbol{k}_b\}$, which initially coincides with a reference vector basis $B_0 = \{\boldsymbol{i}_0, \boldsymbol{j}_0, \boldsymbol{k}_0\}$. After a maneuver, the satellite changes its attitude, and its 3-2-1 Euler angles are $\theta = \psi = \varphi = 90$ deg **(1 point)**.

(a) Express the vectors of B_b in B_0 after the maneuver.
After the maneuver, $\boldsymbol{i}_b = -\boldsymbol{k}_0$, $\boldsymbol{j}_b = \boldsymbol{j}_0$, $\boldsymbol{k}_b = \boldsymbol{i}_0$.

(b) Are there other sets of 3-2-1 Euler angles that result in exactly the same attitude? Justify your answer.

Yes, the configuration incurs in the Euler angle singularity, as the second rotation aligns the first and third rotation axis in space. For example, the set of Euler angles $\theta = 0$ deg, $\psi = 90$ deg, $\varphi = 0$ deg results in the same orientation.

6. Ultima Thule is a trans-Neptunian minor planet of mass m located in the Kuiper belt. It can be modeled as two tangent homogeneous spheres 1 and 2, of diameter d_1 and d_2, respectively. Determine the distance of its center of mass G with respect to point O, and the components of the tensor of inertia about G in the basis B_0 shown **(1.5 points)**.



The volume of each sphere is $V_1 = 4\pi(d_1/2)^3/3$ and $V_2 = 4\pi(d_2/2)^3/3$. The density is $\rho = m/(V_1 + V_2)$. The center of mass is located $x_0^G = \rho V_2(d_1 + d_2)/(2m)$ to the right of O. The moment of inertia of each sphere about an axis that passes by its center is $\rho V_1 d_1^2/10$ and $\rho V_2 d_2^2/10$. With this information, and using Steiner's theorem, the tensor of inertia of Ultima Thule is

$$\overline{\overline{\mathcal{I}}}_G = \frac{\rho(V_1 d_1^2 + V_2 d_2^2)}{10} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_0 + \left[\rho V_1 (x_0^G)^2 + \rho V_2 \left(\frac{d_1 + d_2}{2} - x_0^G \right)^2 \right] \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_0 .$$

7. An airplane with aerodynamic efficiency $E = 15$ is cruising in uniform level flight. At some point the pilot increases the thrust T by 15% while keeping the angle of attack constant to climb to a higher altitude. What is the climb angle γ that results in the steady state? **(1 point)**.

The increment is small, so we may approximate the calculation with the small angle assumption:

$$\gamma = \frac{T - T_{ulf}}{E T_{ulf}} = 0.01 \text{ rad}$$

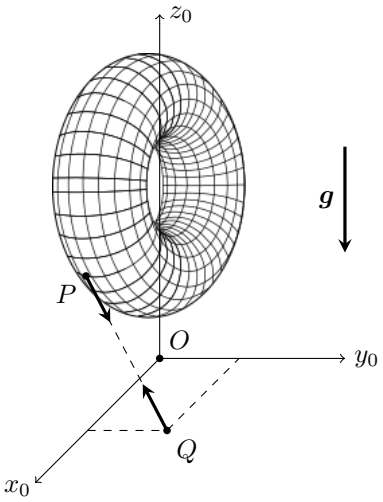
PROBLEM 1

10 points, 65 minutes.

A heavy point particle P with mass $m^P = 1$ moves without friction on a vertically-standing torus as shown in the figure. The particle cannot separate from the surface of this torus. The torus is defined in the Cartesian coordinates of the inertial reference frame S_0 by the following expressions:

$$\begin{cases} x_0 &= (2+\cos\phi)\sin\theta \\ y_0 &= \sin\phi \\ z_0 &= 4+(2+\cos\phi)\cos\theta \end{cases}$$

A second particle Q with mass $m^Q = 2$ is constrained to move on the Ox_0y_0 plane without friction. The two particles P and Q attract each other with a force that is proportional to the distance between them. The proportionality constant of this force is $k = 2$. Initially, the particle P is located at $\boldsymbol{r}_0^P(t_0) = 2\boldsymbol{i}_0 + \boldsymbol{j}_0 + 4\boldsymbol{k}_0$ with $\boldsymbol{v}_0^P(t_0) = -\boldsymbol{k}_0$ and the particle Q is located at $\boldsymbol{r}_0^Q(t_0) = 3\boldsymbol{i}_0 + \boldsymbol{j}_0$ with $\boldsymbol{v}_0^Q(t_0) = -\boldsymbol{i}_0 + \boldsymbol{j}_0$.



1. Pick a minimal, valid set of coordinates to describe the configuration of the system. Indicate what is the geometrical meaning of the chosen coordinates **(1 point)**.

   The system has 4 degrees of freedom, which can be described with the coordinates θ, ϕ (for the position of P) and ξ, η (for the position of Q). The angle θ gives the position of P along the major circle of the torus, while ϕ gives its position along the minor circle. Coordinates ξ and η are merely the x_0, y_0 coordinates of Q.

2. Obtain a general expression for the position, velocity, and acceleration vectors of P and Q in S_0 using the chosen coordinates and their derivatives **(2 points)**.

   The requested expressions for P are:

$$\begin{aligned}
\boldsymbol{r}_0^P &= (2+\cos\phi)\sin\theta\boldsymbol{i}_0 + \sin\phi\boldsymbol{j}_0 + [4+(2+\cos\phi)\cos\theta]\boldsymbol{k}_0 \\
\boldsymbol{v}_0^P &= [-\dot\phi\sin\phi\sin\theta + \dot\theta(2+\cos\phi)\cos\theta]\boldsymbol{i}_0 + \dot\phi\cos\phi\boldsymbol{j}_0 - [\dot\phi\sin\phi\cos\theta + \dot\theta(2+\cos\phi)\sin\theta]\boldsymbol{k}_0 \\
\boldsymbol{a}_0^P &= [-\ddot\phi\sin\phi\sin\theta - \dot\phi^2\cos\phi\sin\theta - 2\dot\phi\dot\theta\sin\phi\cos\theta + \ddot\theta(2+\cos\phi)\cos\theta - \dot\theta^2(2+\cos\phi)\sin\theta]\boldsymbol{i}_0 \\
&\quad + [\ddot\phi\cos\phi - \dot\phi^2\sin\phi]\boldsymbol{j}_0 \\
&\quad - [\ddot\phi\sin\phi\cos\theta + \dot\phi^2\cos\phi\cos\theta - 2\dot\phi\dot\theta\sin\phi\sin\theta + \ddot\theta(2+\cos\phi)\sin\theta + \dot\theta^2(2+\cos\phi)\cos\theta]\boldsymbol{k}_0
\end{aligned}$$

   And for Q:

$$\begin{aligned}
\boldsymbol{r}_0^Q &= \xi\boldsymbol{i}_0 + \eta\boldsymbol{j}_0 \\
\boldsymbol{v}_0^Q &= \dot\xi\boldsymbol{i}_0 + \dot\eta\boldsymbol{j}_0 \\
\boldsymbol{a}_0^Q &= \ddot\xi\boldsymbol{i}_0 + \ddot\eta\boldsymbol{j}_0
\end{aligned}$$

3. Obtain two independent tangent vectors and one normal vector to the torus as a function of the chosen coordinates **(1.5 points)**.

   Two tangent vectors can be obtained by differentiating the parametric equations of the torus with respect to the parameters:

$$\begin{aligned}\boldsymbol{t}_1 &= \cos\theta\boldsymbol{i}_0 - \sin\theta\boldsymbol{k}_0\\ \boldsymbol{t}_2 &= -\sin\phi\sin\theta\boldsymbol{i}_0 + \cos\phi\boldsymbol{j}_0 - \sin\phi\cos\theta\boldsymbol{k}_0\end{aligned}$$

   A normal vector can be obtaining cross-multiplying these, or by first obtaining the implicit expression of the torus:

$$y_0^2 + \left[2 - \sqrt{x_0^2 + (z_0-4)^2}\right]^2 = 1.$$

   And then taking the gradient. After replacing x_0, y_0, z_0 with our coordinates:

$$\boldsymbol{n} = \cos\phi\sin\theta\boldsymbol{i}_0 + \sin\phi\boldsymbol{j}_0 + \cos\phi\cos\theta\boldsymbol{k}_0$$

4. Identify and express mathematically all the forces acting on P and Q **(1.5 points)**.

   The forces acting on P are its weight, the normal reaction of the torus, and the mutual force:

$$\begin{aligned}\boldsymbol{W}_P &= -g\boldsymbol{k}_0\\ \boldsymbol{N}_P &= N_P\cos\phi\sin\theta\boldsymbol{i}_0 + N_P\sin\phi\boldsymbol{j}_0 + N_P\cos\phi\cos\theta\boldsymbol{k}_0\\ \boldsymbol{F}_P &= 2[\xi - (2+\cos\phi)\sin\theta]\boldsymbol{i}_0 + 2[\eta - \sin\phi]\boldsymbol{j}_0 - 2[4 + (2+\cos\phi)\cos\theta]\boldsymbol{k}_0\end{aligned}$$

   where N_P is an unknown magnitude to be computed as part of the solution.

   The forces acting on Q are its weight, the normal reaction of the plane, and the mutual force:

$$\begin{aligned}\boldsymbol{W}_Q &= -2g\boldsymbol{k}_0\\ \boldsymbol{N}_Q &= N_Q\boldsymbol{k}_0\\ \boldsymbol{F}_Q &= -\boldsymbol{F}_P\end{aligned}$$

5. Find down a set of second order differential equations to determine the evolution of the chosen coordinates in time, free of any other unknowns **(1.5 points)**.

   There are four coordinates, so four equations are needed. The first two are obtained by projecting the equation of Newton's second law of P along the tangent directions to the torus. The other two are the equations of Q along $\boldsymbol{i}_0$ and $\boldsymbol{j}_0$. The four equations are:

$$\begin{aligned}-2\dot\phi\dot\theta\sin\phi + \ddot\theta(2+\cos\phi) &= g\sin\theta + 2(\xi\cos\theta + 4\sin\theta)\\ \ddot\phi + \dot\theta^2(2+\cos\phi)\sin\phi &= g\sin\phi\cos\theta + 2[(2 + 4\cos\theta - \xi\sin\theta)\sin\phi + \eta\cos\phi]\\ 2\ddot\xi &= -2[\xi - (2+\cos\phi)\sin\theta]\\ 2\ddot\eta &= -2[\eta - \sin\phi]\end{aligned}$$

6. Obtain a scalar conservation law **(1.5 points)**.

   Considering the system as a whole, the weight and the tension force are conservative, while the normal reactions exert no work. Hence, the total mechanical energy of the system is conserved. The kinetic and potential energy of the system are:

$$T = \frac{1}{2}[\dot{\phi}^2 + \dot{\theta}^2(2+\cos\phi)^2] + [\dot{\xi}^2 + \dot{\eta}^2]$$
$$V = g(2+\cos\phi)\cos\theta + [\xi - (2+\cos\phi)\sin\theta]^2 + [\eta - \sin\phi]^2 + [4 + (2+\cos\phi)\cos\theta]^2$$

   The initial value of T and V is:

$$T(t_0) = \frac{5}{2}$$
$$V(t_0) = 17$$

   and the conservation law reads

$$T + V = T_0 + V_0$$

7. Attending to this conservation law alone, can P ever reach the highest point of the torus? **(1 point)**.

   To check whether this is possible according to the equation of the conservation of the total mechanical energy of the system, we substitute the coordinates for the highest point ($\theta = \phi = 0$) to find whether $T \geq 0$. If this condition holds for some value of ξ, η, then this is an energetically viable configuration of the system. This condition is the least restrictive when $\xi, \eta = 0$:

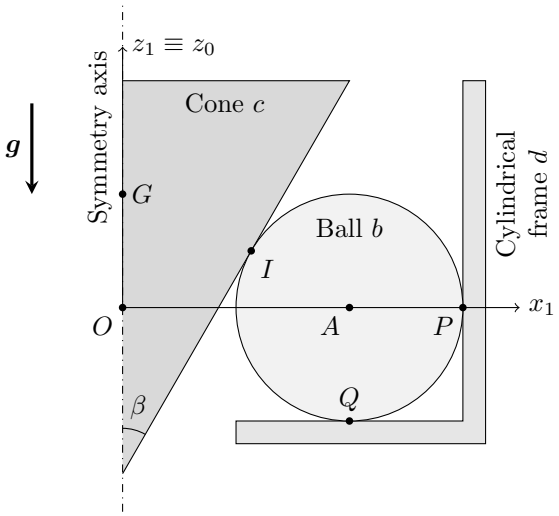
$$3g + 49 \leq \frac{5}{2} + 17.$$

   Regardless of the value of g, the particle cannot reach the top of the torus.

PROBLEM 2

10 points, 75 minutes.

A rigid, homogeneous, heavy ball b of mass m and radius R is always in contact with a rigid, homogeneous, heavy cone c of half-angle β, radius at the base $2R$, and mass $2m$. The ball is also always in contact with a rigid cylindrical frame d of inner radius $3R$ that has a base at the bottom and the same axis of symmetry as the cone. We call the instantaneous contact points between the bodies I, P and Q, as shown in the figure. We call the center of mass of the ball and cone A and G, respectively. The cylinder is at rest in the inertial reference frame S_0. The cone can rotate about its axis, which is constrained to coincide with Oz_0 at all times. We define the auxiliary reference frame S_1 that initially coincides with S_0 and rotates with respect to $\boldsymbol{k}_0$, such that Ox_1 follows the center of the ball in its motion. The angle between Ox_0 and Ox_1 is called θ, and the angle of rotation of the cone with respect to S_0 is called ϕ. Both angles are positive in the counterclockwise direction as seen from Oz_0. The friction coefficient of the ball with the other bodies is μ. The ball can pivot freely at each contact point with the cone and the cylinder.



1. How many effective degrees of freedom does the system have in the limit $\mu = 0$? And in the limit $\mu = \infty$? **(1 point)**.

   In the absence of friction, the ball can rotate freely in all directions and move along the circumference of the cylinder, so it has 4 degrees of freedom. The cone can rotate freely about its axis, bringing the total number of degrees of freedom of the system up to 5.

   In the limit $\mu = \infty$ there is no slipping, and the system only has one effective degree of freedom, associated to the rotation of the cone, which carries the ball along.

2. Write down the general expression of the position, velocity and acceleration of A with respect to S_0 as a function of θ and its derivatives **(1 point)**.

   We begin by defining the vector basis of S_1,

   $$\begin{cases} \boldsymbol{i}_1 = \cos\theta \boldsymbol{i}_0 + \sin\theta \boldsymbol{j}_0 \\ \boldsymbol{j}_1 = -\sin\theta \boldsymbol{i}_0 + \cos\theta \boldsymbol{j}_0 \\ \boldsymbol{k}_1 = \boldsymbol{k}_0 \end{cases} \tag{1}$$

   The expressions sought are:

   $$\boldsymbol{r}_0^A = 2R\boldsymbol{i}_1$$
   $$\boldsymbol{v}_0^A = 2R\dot{\theta}\boldsymbol{j}_1$$
   $$\boldsymbol{a}_0^A = -2R\dot{\theta}^2\boldsymbol{i}_1 + 2R\ddot{\theta}\boldsymbol{j}_1$$

Assuming first that no slipping occurs between the ball and the other objects:

3. Locate the instantaneous axis of rotation (IAR) of the ball in S_0. Write down the general expression of the angular velocity and acceleration of the ball with respect to S_0 as a function of θ and its derivatives **(2 points)**.

   Due to the no-slip condition, points P and Q of the ball have zero velocity, and hence the instantaneous axis of rotation passes through them. This gives the direction of the angular velocity vector:

   $$\boldsymbol{\omega}_{b0} = \lambda(\boldsymbol{i}_1 + \boldsymbol{k}_1)$$

   To relate λ with $\dot{\theta}$, we must use the velocity field expression of the ball:

   $$\boldsymbol{v}_0^A = \boldsymbol{0} + \boldsymbol{\omega}_{b0} \times \boldsymbol{PA}$$

   from where:

   $$\boldsymbol{\omega}_{b0} = -2\dot{\theta}\boldsymbol{i}_1 - 2\dot{\theta}\boldsymbol{k}_1$$

   Finally, taking derivatives,

   $$\boldsymbol{\alpha}_{b0} = -2\ddot{\theta}\boldsymbol{i}_1 - 2\dot{\theta}^2\boldsymbol{j}_1 - 2\ddot{\theta}\boldsymbol{k}_1$$

4. Find the relation between $\dot{\theta}$ and $\dot{\phi}$ **(1 point)**.

   Point I is located at $\boldsymbol{r}_0^I = R(2-\cos\beta)\boldsymbol{i}_1 + R\sin\beta\boldsymbol{k}_1$. Due to the no-slip condition, the velocity of points $I(b)$ and $I(c)$ must be equal:

   $$\begin{aligned}\boldsymbol{v}_0^{I(c)} &= \dot{\phi}R(2-\cos\beta)\boldsymbol{j}_1\\ \boldsymbol{v}_0^{I(b)} &= \boldsymbol{v}_0^A + \boldsymbol{\omega}_{b0}\times\boldsymbol{AI} = 2\dot{\theta}R(1+\cos\beta+\sin\beta)\boldsymbol{j}_1\end{aligned}$$

   Therefore,

   $$\dot{\phi} = 2\dot{\theta}\frac{1+\cos\beta+\sin\beta}{2-\cos\beta} \tag{2}$$

5. Identify and express all forces acting on the ball, and clearly state their point of application **(2 points)**.

   The ball is subject to its own weight (applied on its center of mass A), and the normal and friction reactions at the three contact points (applied on those points):

   $$\begin{aligned}\boldsymbol{W}_b &= -mg\boldsymbol{k}_1\\ \boldsymbol{N}_P &= -N_P\boldsymbol{i}_1\\ \boldsymbol{N}_Q &= N_Q\boldsymbol{k}_1\\ \boldsymbol{N}_{Ib} &= N_I\cos\beta\boldsymbol{i}_1 - N_I\sin\beta\boldsymbol{k}_1\\ \boldsymbol{F}_P &= F_{Py}\boldsymbol{j}_1 + F_{Pz}\boldsymbol{k}_1\\ \boldsymbol{F}_Q &= F_{Qx}\boldsymbol{i}_1 + F_{Qy}\boldsymbol{j}_1\\ \boldsymbol{F}_{Ib} &= F_{I1}\sin\beta\boldsymbol{i}_1 + F_{I2}\boldsymbol{j}_1 + F_{I1}\cos\beta\boldsymbol{k}_1\end{aligned}$$

6. Obtain the equations of motion of the system and prove that the problem is undetermined (i.e., more unknowns than equations) **(2 points)**.

   So far we have introduced two coordinates ϕ and θ, and 9 unknown reactions. Thus, we need 11 equations. These are Newton's and Euler's equations for the ball (6 scalar equations), Newton's and Euler's equation

for the cone along the $\boldsymbol{k}_1$ direction (2 scalar equations), and Eq. (2) (1 scalar equation). Clearly, we are lacking equations to close the problem, so it is hyperstatic. Note that if we had included the reactions that keep the cone on its axis, four more unknowns would exist, but they would have trivial equations (the 4 remaining equations for the cone).

After eliminating ϕ with Eq. (2), the remaining 8 equations are:

$$\begin{aligned}
-2mR\dot{\theta}^2 &= -N_P + N_I\cos\beta + F_{Qx} + F_{I1}\sin\beta \\
2mR\ddot{\theta} &= F_{Py} + F_{Qy} + F_{I2} \\
0 &= -mg + N_Q - N_I\sin\beta + F_{Pz} + F_{I1}\cos\beta \\
-\frac{4mR}{5}\ddot{\theta} &= F_{Qy} - \sin\beta F_{I2} \\
-\frac{4mR}{5}\dot{\theta}^2 &= -F_{Pz} - F_{Qx} + RF_{I1} \\
-\frac{4mR}{5}\ddot{\theta} &= F_{Py} - \cos\beta F_{I2} \\
0 &= -2mg + N_I\sin\beta - F_{I1}\cos\beta \\
\frac{24mR}{5}\frac{1+\cos\beta+\sin\beta}{2-\cos\beta}\ddot{\theta} &= -(2-\cos\beta)F_{I2}
\end{aligned}$$

Considering now that the cone is rotating at a constant angular velocity, and assuming that the component in the Ox_1z_1 plane of the friction forces at points P and I are zero:

7. What is the maximum value of $\dot{\theta}$ for which the no-slipping condition holds? **(1 point)**.

   If $\ddot{\theta} = 0$, and assuming $F_{Pz} = FI1 = 0$, the previous system of equation simplifies and becomes determined:

$$\begin{aligned}
-2mR\dot{\theta}^2 &= -N_P + N_I\cos\beta + F_{Qx} \\
0 &= F_{Py} + F_{Qy} + F_{I2} \\
0 &= -mg + N_Q - N_I\sin\beta \\
0 &= F_{Qy} - \sin\beta F_{I2} \\
-\frac{4mR}{5}\dot{\theta}^2 &= -F_{Qx} \\
0 &= F_{Py} - \cos\beta F_{I2} \\
0 &= -2mg + N_I\sin\beta \\
0 &= -(2-\cos\beta)F_{I2}
\end{aligned}$$

   from where $F_{I2} = F_{Py} = F_{Qy} = 0$, $N_I = 2mg/\sin\beta$, $N_Q = 3mg$. Slipping occurs beyond the angular rate at which the friction force F_{Qx} is maximal,

$$-\frac{4mR\dot{\theta}^2}{5} = -3\mu mg \Rightarrow \dot{\theta}^2 = \frac{15\mu g}{4R}$$

Note: The moment of inertia of the sphere about any axis that passes through its center is $2mR^2/5$. The moment of inertia of the cone about its axis is $12mR^2/5$.