SHORT QUESTIONS

10 points, 75 minutes.

Please answer the questions below. Questions requiring a True/False answer also require a justification of your answer. **A right answer without explanation will not be graded**.

1. A point particle P describes a 3D trajectory given by the following equations in cylindrical coordinates as a function of time:

$$\begin{cases} R &= 2 \\ \theta &= \sin(t^2) \\ z &= 2\cos(t^2) \end{cases} \tag{1}$$

Describe the trajectory of the particle. Write down the tangent, normal, and binormal unit vectors $\{\boldsymbol{e}_t, \boldsymbol{e}_n, \boldsymbol{e}_b\}$ as function of t, in terms of the cylindrical vector basis. Express the tangential and normal acceleration of the particle as function of t **(1.5 points)**.

The trajectory is the intersection of two cylinders, $x^2 + z^2 = 4$ and $x^2 + y^2 = 4$. The unit vectors are:

$$\boldsymbol{e}_t = \cos(t^2)\boldsymbol{e}_\theta - \sin(t^2)\boldsymbol{e}_z \tag{2}$$

$$\boldsymbol{e}_n = -\frac{\cos^2(t^2)\boldsymbol{e}_R + \sin(t^2)\boldsymbol{e}_\theta + \cos(t^2)\boldsymbol{e}_z}{\sqrt{1+\cos^4(t^2)}} \tag{3}$$

$$\boldsymbol{e}_b = \frac{-\boldsymbol{e}_R + \sin(t^2)\cos^2(t^2)\boldsymbol{e}_\theta + \cos^3(t^2)\boldsymbol{e}_z}{\sqrt{1+\cos^4(t^2)}} \tag{4}$$

The acceleration is $\boldsymbol{a}_0^P = -8t^2\cos^2(t^2)\boldsymbol{e}_R + [4\cos(t^2) - 8t^2\sin(t^2)]\boldsymbol{e}_\theta - [4\sin(t^2) + 8t^2\cos(t^2)]\boldsymbol{e}_z$. Projecting along $\boldsymbol{e}_t$ and $\boldsymbol{e}_n$ we obtain the tangential and normal acceleration:

$$a_{0t}^P = 4 \tag{5}$$

$$a_{0n}^P = -8t^2\cos^2(t^2)\boldsymbol{e}_R - 8t^2\sin(t^2)\boldsymbol{e}_\theta - 8t^2\cos(t^2)\boldsymbol{e}_z. \tag{6}$$

2. A point particle P of unit mass moves in the one-dimensional potential given by $V(x) = x^2\exp(x)$. Indicate which of the following statements are true or false and provide a brief justification **(1.5 points)**.

   (a) If P is released at $x = -1$ with zero velocity, the particle will start to move toward the right and its motion will remain bounded for all times.
   The potential has a minimum $V = 0$ at $x = 0$, a maximum $V = 4\exp(-2)$ at $x = -2$, goes to $V \to 0$ as $x \to -\infty$, and diverges to $V \to \infty$ as $x \to \infty$. Since $V(-1) < V(-2)$ and $\mathrm{d}V/\mathrm{d}x < 0$ at $x = -1$, the particle is trapped in the potential well that exists around $x = 0$ and will move to the right first. TRUE.

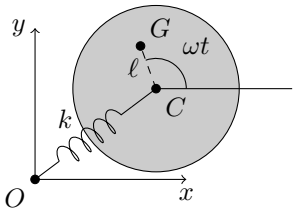
   (b) If P is found at $x = 2$ at some instant of time, it is certain that the particle will end up escaping to $x \to -\infty$.
   $V(2) > V(-2)$, so the particle has enough energy to overcome the maximum of the potential. TRUE.

   (c) If for whatever reason P escapes to $x \to -\infty$, it accelerates to infinite speed.
   FALSE. $T = E - V$; since E must be finite and $V \to 0$ as $x \to -\infty$, T is finite, and so is the velocity.

   (d) The particle can remain at rest at $x = -2$.
   TRUE. The critical points of the potential are equilibrium points. Maxima are unstable, but equilibrium points nonetheless.

3. To understand the vibrations that occur in a turbojet when the mass distribution of the rotor is slightly unbalanced, we can consider the turboject as a rigid disk of mass m that rotates at a fixed angular velocity ω, with the center of mass G being separated from the geometric center C by a fixed distance. An elastic restoring force (with constant k) tries to keep C at its design position O. We call x_0^G, y_0^G the Cartesian coordinates of G, and x_0^C, y_0^C the Cartesian coordinates of C.

Considering the weight of the disk negligible, write down the equation of the dynamics of G. Assuming that the homogeneous solution is damped over time and only the particular solution remains, write down the expression of $x_0^G(t)$ and $x_0^C(t)$. Can we affirm that $|\boldsymbol{OC}| > |\boldsymbol{OG}|$ if $\omega > \sqrt{k/m}$? **(1.5 point)**.



The equation of dynamics of G yields:

$$m\ddot{x}_0^G = -kx_0^C = -k[x_0^G - \ell\cos(\omega t)] \tag{7}$$
$$m\ddot{y}_0^G = -ky_0^C = -k[y_0^G - \ell\sin(\omega t)] \tag{8}$$

Rearranging the x_0^G equation:

$$\ddot{x}_0^G + \omega_0^2 x_0^G = \omega_0^2 \ell\cos(\omega t) \tag{9}$$

where $\omega_0 = \sqrt{k/m}$. As long as $\omega \neq \omega_0$, the particular solution to this equation is

$$x_0^G(t) = \frac{\omega_0^2\ell}{\omega_0^2 - \omega^2}\cos(\omega t); \quad x_0^C(t) = \frac{\omega^2\ell}{\omega_0^2 - \omega^2}\cos(\omega t) \tag{10}$$

As a result, we can obtain $|\boldsymbol{OG}| = \omega_0^2\ell/(\omega_0^2 - \omega^2)$ and $|\boldsymbol{OC}| = \omega^2\ell/(\omega_0^2 - \omega^2)$. Clearly, when $\omega > \omega_0$, $|\boldsymbol{OC}| > |\boldsymbol{OG}|$, so the statement is true.

4. A spacecraft is orbiting a planet that has a gravitational parameter μ. Its orbit is circular, with a radius $r = a$. We want the spacecraft to change into a escape trajectory to leave the planet and reach infinity ($r = \infty$), so we fire its rockets to provide an instantaneous change of velocity, Δv. What is the minimum Δv that we need to achieve this maneuver? **(0.75 point)**.

   The velocity of the spacecraft in its circular orbit is $v_c = \sqrt{\mu/a}$. To set it into an escape trajectory, its velocity must be at least $v_e = \sqrt{2\mu/a}$. Hence, the change of velocity that must be applied is $\Delta v = v_e - v_c = \sqrt{\mu/a}(\sqrt{2} - 1)$.

5. Three simultaneous measurements are made of the motion of a rigid airplane. The center of mass G, located at $\boldsymbol{r}_0^G = \boldsymbol{0}$ [m], has a velocity $\boldsymbol{v}_0^G = 4\boldsymbol{i} + 2\boldsymbol{j}$ [m/s]. Point A on the wing has $\boldsymbol{r}_0^A = -\boldsymbol{i} + 2\boldsymbol{j}$ [m] and $\boldsymbol{v}_0^A = 2\boldsymbol{i} + \boldsymbol{j} + 3\boldsymbol{k}$ [m/s]. Point C on the vertical tail has $\boldsymbol{r}_0^C = -4\boldsymbol{i} + \boldsymbol{k}$ [m] and $\boldsymbol{v}_0^C = 5\boldsymbol{i} - 3\boldsymbol{j} + 4\boldsymbol{k}$ [m/s]. What is the minimal velocity that the points of airplane have at that instant? **(1.5 points)**.

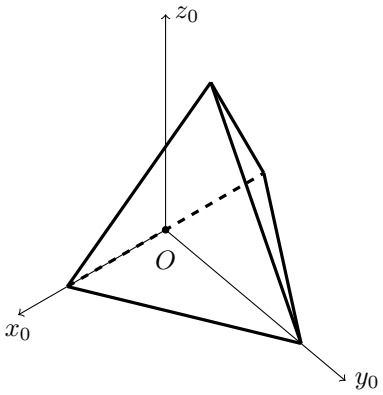
   We can use two independent equations of the velocity field of the body to compute $\boldsymbol{\omega}_{B0}$:

$$\boldsymbol{v}_0^A - \boldsymbol{v}_0^G = \boldsymbol{\omega}_{B0} \times \boldsymbol{r}_0^A \tag{11}$$
$$\boldsymbol{v}_0^C - \boldsymbol{v}_0^G = \boldsymbol{\omega}_{B0} \times \boldsymbol{r}_0^C \tag{12}$$

   It is easy to see that of these six scalar equations, three of them yield $\boldsymbol{\omega}_{B0} = \boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$ rad/s, and the other three are redundant and compatible. The minimum possible velocity in the rigid body is

$$\boldsymbol{v}_{0\parallel} = \frac{\boldsymbol{v}^G \cdot \boldsymbol{\omega}_{B0}}{\omega_{B0}^2}\boldsymbol{\omega}_{B0} = 2\boldsymbol{i} + 2\boldsymbol{j} + 2\boldsymbol{k} \text{ m/s}. \tag{13}$$

6. Knowing that the moment of inertia of a regular, homogeneous tetrahedron of mass m and edge length a about *any* axis that passes by its geometric centroid is equal to $ma^2/20$, compute the tensor of inertia in the reference frame S_0 shown in the figure **(1.5 points)**.



The height of one face is $a\sqrt{3}/2$; the height of the tetrahedron is $a\sqrt{6}/3$. The vector that goes from O to the centroid C of the tetrahedron is:

$$\boldsymbol{OC} = \frac{a\sqrt{3}}{6}\boldsymbol{j}_0 + \frac{a\sqrt{6}}{12}\boldsymbol{k}_0 \tag{14}$$

From the information of the problem, the tensor of inertia of the tetrahedron about C is diagonal, with:

$$\bar{\bar{I}}_C = \frac{ma^2}{20}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{15}$$

Direct application of Steiner's theorem then gives:

$$\bar{\bar{I}}_O = \bar{\bar{I}}_C + \frac{ma^2}{8}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{ma^2}{24}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & \sqrt{2} \\ 0 & \sqrt{2} & 1 \end{bmatrix} = \frac{ma^2}{120}\begin{bmatrix} 21 & 0 & 0 \\ 0 & 11 & -5\sqrt{2} \\ 0 & -5\sqrt{2} & 16 \end{bmatrix} \tag{16}$$

7. Two rocket systems have been mounted on a rigid asteroid, and they fire simultaneously to create a system of forces applied on points of the body. In dimensionless units, point A of the asteroid at $\boldsymbol{r}_0^A = \boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$ experiences a force $\boldsymbol{F}_A = -4\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$; and point B at $\boldsymbol{r}_0^B = -2\boldsymbol{i} + 3\boldsymbol{j} + 2\boldsymbol{k}$ has $\boldsymbol{F}_B = 8\boldsymbol{i} - 6\boldsymbol{j} - 5\boldsymbol{k}$. If we know that the resultant moment of forces about the center of mass G of the asteroid is zero, and that the center of mass is located in the plane $2x - 2y - 3z = 1/2$, find G **(1 point)**.

The resultant force is $\boldsymbol{F} = 4\boldsymbol{i} - 4\boldsymbol{j} - 6\boldsymbol{k}$, and the resultant moment of forces about the origin O is $\boldsymbol{M}_O = -6\boldsymbol{i} + 3\boldsymbol{j} - 6\boldsymbol{k}$. The locus of points that have minimal resultant moment of forces is found by using the relation

$$\boldsymbol{M}_P = \boldsymbol{M}_O + \boldsymbol{F} \times \boldsymbol{r}_0^P \tag{17}$$

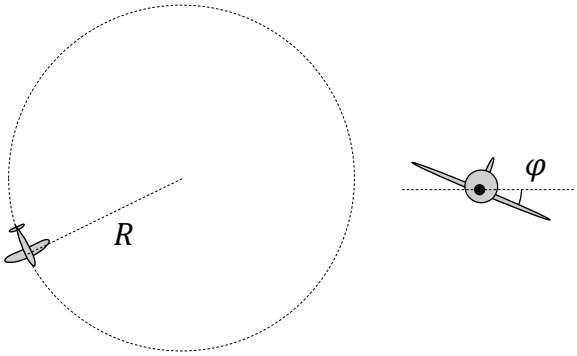
by cross multiplying on the left by $\boldsymbol{F}$ and imposing that $\boldsymbol{M}_P = \boldsymbol{0}$:

$$\boldsymbol{r}_0^G = \frac{\boldsymbol{F} \times \boldsymbol{M}_O}{F^2} + \lambda\boldsymbol{F} = \frac{42\boldsymbol{i} + 60\boldsymbol{j} - 12\boldsymbol{k}}{68} + \lambda(4\boldsymbol{i} - 4\boldsymbol{j} - 6\boldsymbol{k}) \tag{18}$$

Imposing the additional condition for the position of G, we find that $\lambda = 1/68$, so

$$\boldsymbol{r}_0^G = \frac{46\boldsymbol{i} + 56\boldsymbol{j} - 18\boldsymbol{k}}{68} = \frac{23\boldsymbol{i} + 28\boldsymbol{j} - 9\boldsymbol{k}}{34} \tag{19}$$

8. A heavy airplane performs a coordinated horizontal turn of radius R at a flight velocity u_∞. Compute the required roll angle φ and the load factor n **(0.75 points)**.

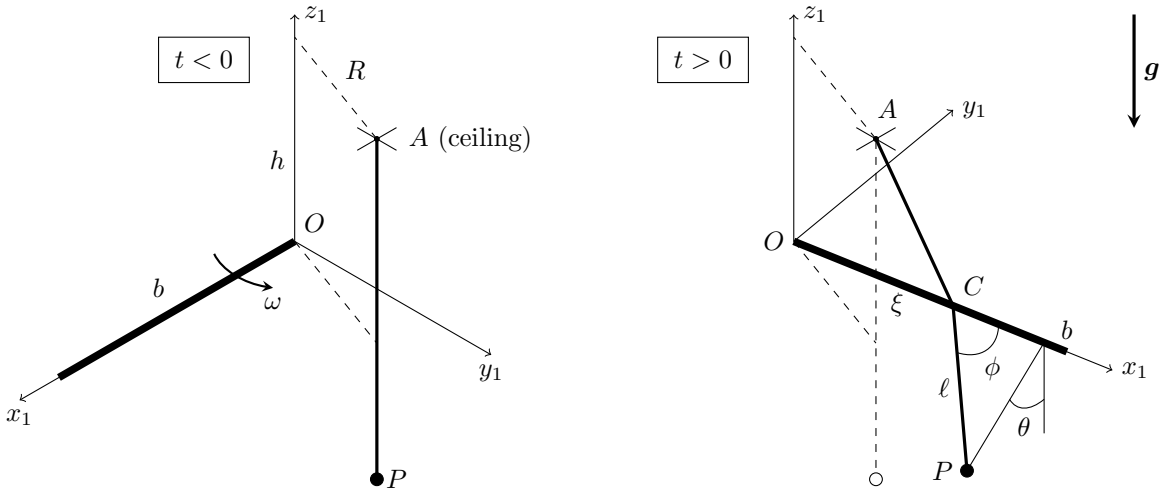


$$\tan\varphi = u_\infty^2/(gR); \quad n = 1/\cos\varphi. \tag{20}$$

PROBLEM 1

10 points, 75 minutes.

A heavy point particle P with mass m hanging from a fixed point A on the ceiling with a massless string of constant length L. The particle is initially at rest. A bar b rotates about its center O in a horizontal plane at a distance h from the ceiling, with a known, constant angular velocity ω. The horizontal distance between O and A is R. At time $t = 0$, the bar collides with the string and drags it along its motion. The string slides over the bar without friction. We call C the instantaneous point of contact of the string with the bar. We define a reference frame S_1 that rotates with the bar as shown in the figure. We call ξ the distance between O and C; ℓ the length of the string segment between C and P; θ the angle between the vertical plane Ox_1z_1 and the plane defined by the bar and P; and ϕ the angle formed between the string segment $\boldsymbol{CP}$ and the bar. You are asked to study only the part of the motion in which the string remains tense and in contact with the bar.



1. Write down the position and velocity of point A in S_1 as a function of time **(0.5 points)**.

   As seen from the rotating reference frame S_1, A seems to be counter-rotating:

$$\boldsymbol{r}_1^A = R\cos(\omega t)\boldsymbol{i}_1 - R\sin(\omega t)\boldsymbol{j}_1 + h\boldsymbol{k}_1,$$
$$\boldsymbol{v}_1^A = -R\omega\sin(\omega t)\boldsymbol{i}_1 - R\omega\cos(\omega t)\boldsymbol{j}_1.$$

2. Obtain a general expression for the position, velocity, and acceleration of P in S_1 as a function of ξ, ℓ θ, ϕ, their derivatives, and time **(3 points)**.

   We define the auxiliary vector basis B_2:

$$\begin{cases} \boldsymbol{i}_2 = \boldsymbol{i}_1 \\ \boldsymbol{j}_2 = \cos\theta\boldsymbol{j}_1 - \sin\theta\boldsymbol{k}_1 \\ \boldsymbol{k}_2 = \sin\theta\boldsymbol{j}_1 + \cos\theta\boldsymbol{k}_1 \end{cases} \tag{21}$$

   Then, for the point particle P:

$$\boldsymbol{r}_1^P = \boldsymbol{r}_1^C + \boldsymbol{CP} = (\xi + \ell\cos\phi)\boldsymbol{i}_2 - \ell\sin\phi\boldsymbol{k}_2,$$
$$\boldsymbol{v}_1^P = (\dot{\xi} + \dot{\ell}\cos\phi - \ell\dot{\phi}\sin\phi)\boldsymbol{i}_2 - \ell\dot{\theta}\sin\phi\boldsymbol{j}_2 - (\dot{\ell}\sin\phi + \ell\dot{\phi}\cos\phi)\boldsymbol{k}_2$$
$$\begin{aligned}\boldsymbol{a}_1^P = {} & (\ddot{\xi} + \ddot{\ell}\cos\phi - \ell\ddot{\phi}\sin\phi - 2\dot{\ell}\dot{\phi}\sin\phi - \ell\dot{\phi}^2\cos\phi)\boldsymbol{i}_2 \\ & - (2\dot{\ell}\dot{\theta}\sin\phi + \ell\ddot{\theta}\sin\phi + 2\ell\dot{\theta}\dot{\phi}\cos\phi)\boldsymbol{j}_2 \\ & - [\ddot{\ell}\sin\phi + 2\dot{\ell}\dot{\phi}\cos\phi + \ell\ddot{\phi}\cos\phi - \ell(\dot{\phi}^2 + \dot{\theta}^2)\sin\phi]\boldsymbol{k}_2\end{aligned}$$

   Note that angles θ and ϕ can be regarded as spherical angles for the position of P as seen from C.

3. Express $\boldsymbol{AC}$, ℓ , and ϕ as a function of of ξ and time **(1.5 points)**.

   In the case of ℓ we have:

$$\ell = L - \sqrt{R^2 + \xi^2 - 2R\xi\cos(\omega t) + h^2} \tag{22}$$

   Since the string can slide on the bar without friction, the angle between the segment $\boldsymbol{CA}$ and the bar is also ϕ. Hence we can write:

$$\cos\phi = \frac{\xi - R\cos(\omega t)}{L - \ell} \tag{23}$$

4. Identify and express all the forces acting on P **(1.5 points)** .

   The problem has two effective degrees of freedom, which we can identify with coordinates ϕ, θ. An additional unknown T appears when writing the tension force on the string.

   In preparation for writing the equation of dynamics in S_1, which rotates with angular velocity $\boldsymbol{\omega}_{10} = \omega\boldsymbol{k}_1$ relative to inertial space, the forces acting on the particle are its weight, the string tension, the centrifugal force, and the Coriolis force:

$$\begin{aligned}
\boldsymbol{W} &= -mg\boldsymbol{k}_1 \\
\boldsymbol{T} &= -T\cos\phi\boldsymbol{i}_1 + T\sin\phi\sin\theta\boldsymbol{j}_1 + T\sin\phi\cos\theta\boldsymbol{k}_1 \\
\boldsymbol{F}_{\text{cent}} &= m\omega^2(\xi + \ell\cos\phi)\boldsymbol{i}_1 - m\omega^2\ell\sin\phi\sin\theta\boldsymbol{j}_1 \\
\boldsymbol{F}_{\text{Cori}} &= -2m\omega(\dot\ell\sin\phi\sin\theta + \ell\dot\phi\cos\phi\sin\theta + \ell\dot\theta\sin\phi\cos\theta)\boldsymbol{i}_1 - 2m\omega(\dot\xi + \dot\ell\cos\phi - \ell\dot\phi\sin\phi)\boldsymbol{j}_1
\end{aligned}$$

5. Write down a set of second order differential equations for the evolution of the unknowns of the problem **(1 points)**.

   The problem has two effective degrees of freedom, which we can identify with coordinates ϕ, θ. The unknowns of the problem are ϕ, θ, ℓ, ξ, and T. The three equations needed result from projecting Newton's second law along the vectors of the basis B_1:

$$\begin{aligned}
&(\ddot\xi + \ddot\ell\cos\phi - \ell\ddot\phi\sin\phi - 2\dot\ell\dot\phi\sin\phi - \ell\dot\phi^2\cos\phi) = \\
&\quad \omega^2(\xi + \ell\cos\phi) - 2\omega(\dot\ell\sin\phi\sin\theta + \ell\dot\phi\cos\phi\sin\theta + \ell\dot\theta\sin\phi\cos\theta) - (T/m)\cos\phi \\
&[\sin\phi\sin\theta(\ddot\ell - \ell\dot\phi^2 - \ell\dot\theta^2) + \sin\phi\cos\theta(\ell\ddot\theta + 2\dot\ell\dot\theta) + 2\ell\dot\phi\dot\theta\cos\phi\cos\theta + \cos\phi\sin\theta(\ell\ddot\phi + 2\dot\ell\dot\phi)] = \\
&\quad (T/m)\sin\phi\sin\theta - \omega^2\ell\sin\phi\sin\theta - 2\omega(\dot\xi + \dot\ell\cos\phi - \ell\dot\phi\sin\phi) \\
&[\sin\phi\cos\theta(\ddot\ell - \ell\dot\phi^2 - \ell\dot\theta^2) - \sin\phi\sin\theta(\ell\ddot\theta + 2\dot\ell\dot\theta) - 2\ell\dot\phi\dot\theta\cos\phi\sin\theta + \cos\phi\cos\theta(\ell\ddot\phi + 2\dot\ell\dot\phi)] = \\
&\quad (T/m)\sin\phi\cos\theta - g
\end{aligned}$$

6. Compute the initial conditions for the previous equations at $t = 0$ **(1 point)**.

   At $t = 0$, the string is a straight line so $\boldsymbol{r}^P_{1,0} = -(L-h)\boldsymbol{k}_1$, and P is moving with a velocity $\boldsymbol{v}^P_1 = \boldsymbol{v}^A_1 = -R\omega\boldsymbol{j}_1$. Comparing these values with the general expressions for $\boldsymbol{r}^P_1$ and $\boldsymbol{v}^P_1$, we obtain the initial conditions for the two DOF that we have identifed:

$$\begin{aligned}
&\phi_0 = \pi/2; && \theta_0 = 0; \\
&\dot\phi_0 = 0; && \dot\theta_0 = R\omega/(L-h).
\end{aligned}$$

7. Is the mechanical energy of the particle conserved in time? Justify your answer **(0.5 points)**.

   No, it is not conserved, neither in S_1 nor in an inertial reference frame. The reason is that the non-conservative tension force $\boldsymbol{T}$ exerts work on the particle.

8. If/when the string separates from the bar, express the geometric condition as a relation between θ and and time **(1 point)**.

   The angle β that the plane OCA makes with the vertical plane Ox_1z_1 is

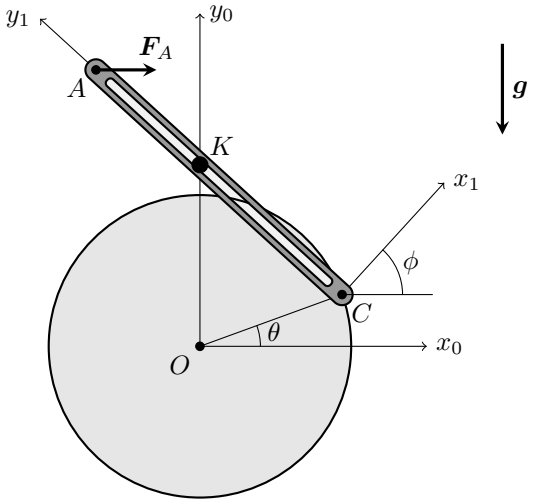
$$\tan\beta = \frac{R\sin(\omega t)}{h} \tag{24}$$

   The string separates from the bar if $\theta + \beta = 0$, i.e., if the string becomes fully straight.

PROBLEM 2

10 points, 60 minutes.

A homogeneous disk D of radius R and mass m can rotate freely about its center O, which is fixed to the wall. We define the inertial reference frame S_0 as shown in the figure. A homogeneous, heavy bar B of mass $2m$ and length L has a longitudinal groove that allows it to slide without friction over a small knob K fixed to the wall and located at a height h above O. One of the end points of the bar is connected to perimeter of the disk at point C as shown. This connection allows the bar to rotate freely about C with respect to the disk. The other end point A of the bar is attracted to the Oy axis by a force $\boldsymbol{F}_A$ that is proportional to its distance to Oy. The proportionality constant of this force is k.



1. How many effective degrees of freedom does this planar problem have? Justify your answer **(0.5 points)**.

   Only one: The disk center is attached to the wall so it can only rotate (1); the bar is completely slave to the motion of the disk.

2. Find the geometric relation between θ and ϕ. Compute $\dot{\phi}$ and $\ddot{\phi}$ **(1.5 points)**.

   Since the angle OKC is equal to ϕ, we have

   $$R\sin\theta + |KC|\cos\phi = h \ , \quad KC\sin\phi = R\sin\theta \ ,$$

   which leads to

   $$\phi = \arctan\frac{R\cos\theta}{h - R\cos\theta} \quad \text{for} \quad -\pi/2 < \phi < \pi/2 \ .$$

   The time derivatives of ϕ then read

   $$\dot{\phi} = \frac{R\dot{\theta}(R - h\sin\theta)}{R^2 + h^2 - 2Rh\sin\theta}$$
   $$\ddot{\phi} = \frac{R\ddot{\theta}(R - h\sin\theta)(R^2 + h^2 - 2Rh\sin\theta) + Rh\dot{\theta}^2\cos\theta(R^2 - h^2)}{(R^2 + h^2 - 2Rh\sin\theta)^2} \ .$$

3. Express the general position, velocity, and acceleration of point A and the center of mass of the bar G as a function of θ, ϕ and their time derivatives **(2.5 points)**.

   It is helpful to define $\boldsymbol{i}_1 = \cos\phi\boldsymbol{i}_0 + \sin\phi\boldsymbol{j}_0$, $\boldsymbol{j}_1 = -\sin\phi\boldsymbol{i}_0 + \cos\phi\boldsymbol{j}_0$ and $\boldsymbol{i}_2 = \cos\theta\boldsymbol{i}_0 + \sin\theta\boldsymbol{j}_0$, $\boldsymbol{j}_2 =$

$-\sin\theta\boldsymbol{i}_0 + \cos\theta\boldsymbol{j}_0$. Then:

$$\boldsymbol{r}_0^C = R\boldsymbol{i}_2 \tag{25}$$
$$= R\cos\theta\boldsymbol{i}_0 + R\sin\theta\boldsymbol{j}_0 \tag{26}$$
$$\boldsymbol{r}_0^A = R\boldsymbol{i}_2 + L\boldsymbol{j}_1 \tag{27}$$
$$= (R\cos\theta - L\sin\phi)\boldsymbol{i}_0 + (R\sin\theta + L\cos\phi)\boldsymbol{j}_0 \tag{28}$$
$$\boldsymbol{v}_0^A = R\dot{\theta}\boldsymbol{j}_2 - L\dot{\phi}\boldsymbol{i}_1 \tag{29}$$
$$= -(R\dot{\theta}\sin\theta + L\dot{\phi}\cos\phi)\boldsymbol{i}_0 + (R\dot{\theta}\cos\theta - L\dot{\phi}\sin\phi)\boldsymbol{j}_0 \tag{30}$$
$$\boldsymbol{a}_0^A = -R\dot{\theta}^2\boldsymbol{i}_2 + R\ddot{\theta}\boldsymbol{j}_2 - L\ddot{\phi}\boldsymbol{i}_1 - L\dot{\phi}^2\boldsymbol{j}_1 \tag{31}$$
$$= -(R\ddot{\theta}\sin\theta + R\dot{\theta}^2\cos\theta + L\ddot{\phi}\cos\phi - L\dot{\phi}^2\sin\phi)\boldsymbol{i}_0 \tag{32}$$
$$+ (R\ddot{\theta}\cos\theta - R\dot{\theta}^2\sin\theta - L\ddot{\phi}\sin\phi - L\dot{\phi}^2\cos\phi)\boldsymbol{j}_0 \tag{33}$$

Point G is midway between C and A, so

$$\boldsymbol{r}_0^G = R\boldsymbol{i}_2 + (L/2)\boldsymbol{j}_1 \tag{34}$$
$$= [R\cos\theta - (L/2)\sin\phi]\boldsymbol{i}_0 + [R\sin\theta + (L/2)\cos\phi]\boldsymbol{j}_0 \tag{35}$$
$$\boldsymbol{v}_0^G = R\dot{\theta}\boldsymbol{j}_2 - (L/2)\dot{\phi}\boldsymbol{i}_1 \tag{36}$$
$$= -[R\dot{\theta}\sin\theta + (L/2)\dot{\phi}\cos\phi]\boldsymbol{i}_0 + [R\dot{\theta}\cos\theta - (L/2)\dot{\phi}\sin\phi]\boldsymbol{j}_0 \tag{37}$$
$$\boldsymbol{a}_0^G = -R\dot{\theta}^2\boldsymbol{i}_2 + R\ddot{\theta}\boldsymbol{j}_2 - (L/2)\ddot{\phi}\boldsymbol{i}_1 - (L/2)\dot{\phi}^2\boldsymbol{j}_1 \tag{38}$$
$$= -[R\ddot{\theta}\sin\theta + R\dot{\theta}^2\cos\theta + (L/2)\ddot{\phi}\cos\phi - (L/2)\dot{\phi}^2\sin\phi]\boldsymbol{i}_0 \tag{39}$$
$$+ [R\ddot{\theta}\cos\theta - R\dot{\theta}^2\sin\theta - (L/2)\ddot{\phi}\sin\phi - (L/2)\dot{\phi}^2\cos\phi]\boldsymbol{j}_0 \tag{40}$$

4. Write down the general expression of all forces acting on the bar and on the disk, be they applied or otherwise **(2 points)**.

   The forces on the bar are:

   (a) Its weight $\boldsymbol{W}_B = -2mg\boldsymbol{j}_0$ acting on G.
   (b) The attractive force $\boldsymbol{F}_A = -k(R\cos\theta - L\sin\phi)\boldsymbol{i}_0$ acting on A.
   (c) The reaction at K, $\boldsymbol{F}_K = \lambda_1\boldsymbol{i}_1 = \lambda_1\cos\phi\boldsymbol{i}_0 + \lambda_1\sin\phi\boldsymbol{j}_0$.
   (d) The reaction at $C(B)$, $\boldsymbol{F}_{C(B)} = \lambda_2\boldsymbol{i}_0 + \lambda_3\boldsymbol{j}_0$.

   The forces acting on the disk are:

   (a) Its weight $\boldsymbol{W}_D = -mg\boldsymbol{j}_0$ acting on O.
   (b) The reaction at $C(D)$, $\boldsymbol{F}_{C(D)} = -\lambda_2\boldsymbol{i}_0 - \lambda_3\boldsymbol{j}_0$.
   (c) The reaction at O, $\boldsymbol{F}_O = \lambda_4\boldsymbol{i}_0 + \lambda_5\boldsymbol{j}_0$.

5. Compute the general expression of the kinetic energy of the bar and the kinetic energy of the disk **(2 points)**.

   The moment of inertia in the $\boldsymbol{k}_0$ direction of the bar about G is $mL^2/6$, and of the disk about O is $mR^2/2$. Hence,

$$T^B = m\left[R^2\dot{\theta}^2 + \frac{L^2\dot{\phi}^2}{4} + RL\dot{\theta}\dot{\phi}\sin(\theta-\phi)\right] + \frac{mL^2\dot{\phi}^2}{12} = m\left[R^2\dot{\theta}^2 + \frac{L^2\dot{\phi}^2}{3} + RL\dot{\theta}\dot{\phi}\sin(\theta-\phi)\right] \tag{41}$$
$$T^D = \frac{mR^2\dot{\theta}^2}{4} \tag{42}$$

6. If $\theta_0 = 0$, $\dot{\theta}_0 = 0$ initially, obtain one equation on θ, ϕ, $\dot{\theta}$, $\dot{\phi}$ to describe the motion of the system that does not depend on any additional unknowns **(1.5 points)**.

   If we consider the disk + bar system as our system of analysis, the reaction forces at C are internal. The reaction forces at O and K do not work, since the velocity of the corresponding point of the system is either zero or orthogonal to the reaction force. The gravity force on the disk exerts no work either, since its center of mass does not move. Lastly, the gravity force on the bar and the attractive force are conservative and have the following potential forms:

$$V_g = 2mg[R\sin\theta + (L/2)\cos\phi] \tag{43}$$

$$V_{F_A} = \frac{1}{2}k(R\cos\theta - L\sin\phi)^2 \tag{44}$$

   Hence, the total mechanical energy of the system is conserved:

$$E = m\left[\frac{5R^2\dot{\theta}^2}{4} + \frac{L^2\dot{\phi}^2}{3} + RL\dot{\theta}\dot{\phi}\sin(\theta-\phi)\right] + 2mg\left[R\sin\theta + \frac{L}{2}\cos\phi\right] + \frac{1}{2}k(R\cos\theta - L\sin\phi)^2 \tag{45}$$

   Using the initial conditions given (and noting that $\tan\phi_0 = R/h$), the value of the constant E can be computed:

$$E = \frac{mgL}{\sqrt{R^2/h^2+1}} + \frac{1}{2}kR^2\left(1 - \frac{L/h}{\sqrt{R^2/h^2+1}}\right)^2 \tag{46}$$