# THEORY

*10 points, 50 minutes.*

Please answer the questions 1-6 with True or False and provide a short explanation for your answer. **A right answer without explanation will not be considered.** Solve also exercise 7.

1. Consider two identical gliders, with the same weight, $W$, wing surface, $S$ and aerodynamic coefficients for lift, $c_L$, and drag, $c_D$. Recall that the aerodynamic forces are given by the expressions $L = 0.5\rho V^2 S c_L$ and $D = 0.5\rho V^2 S c_D$, where $L$ is the lift, $D$ is the drag, $V$ is the flight speed and $\rho$ is the air density. Assume that the time interval considered is small so that $\rho$ can be assumed to be constant. The fuselage of glider 2 is slightly damaged. This fact does not affect the lift coefficient. However, the drag coefficient is increased by a factor of 10%, so that its value is now $1.1c_D$. If both gliders are descending at constant speed, and assuming that the descent angle is very small in both cases, then

   (a) the descent angle of glider 2 is 10% larger than the descent angle of glider 1 (True/False) **0.75 points**
   This statement is true. Since the gliders are descending at constant speed in rectilinear motion, the acceleration is zero, then Newton's second law results in

$$\begin{aligned} L &= W\cos\gamma \simeq W, \\ D &= W\sin\gamma \simeq W\gamma, \\ \gamma_1 &= D/L = c_D/c_L, \\ \gamma_2 &= D/L = 1.1c_D/c_L = 1.1\gamma_1. \end{aligned}$$

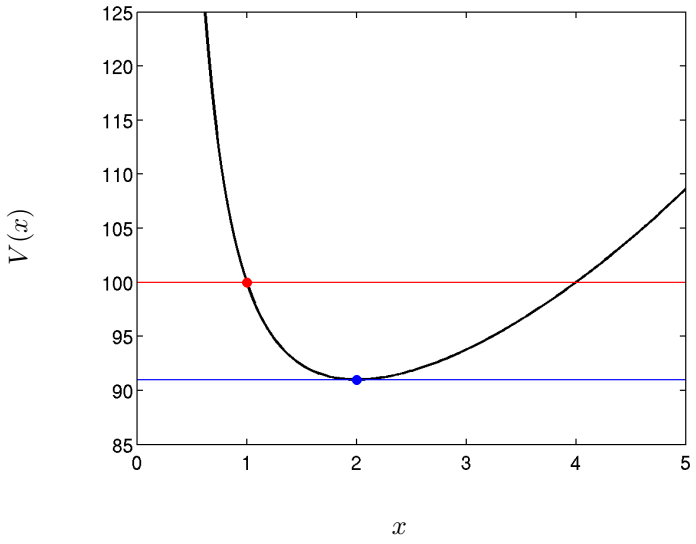
   (b) the speed of glider 1 is 10% larger than the speed of glider 2 (True/False) **0.75 points**
   This statement is false. Since $L = W$, $L = 0.5\rho V^2 S c_L$ and the lift coefficient, $c_L$, is unaffected by the fuselage damage, both gliders descend at the same speed.

2. A point particle is moving along a known path, $\boldsymbol{r_1}(s)$, which is a vector function of the arc length coordinate $s$. Since the arc length coordinate is a function of time $s(t)$, then at each position along the path, the direction of the tangent and normal vectors to the path will depend on how fast the particle is moving along the path. (True/False) **1 point**

   This statement is false. The tangent and normal vectors are purely geometrical entities that do not change once the path is fixed.

3. A point particle is constrained to move along the positive part of the $x-$axis under the potential

$$V(x) = x^2 + \frac{16}{x^2} + 83$$



(a) If initially the particle is at rest at $x = 2$, then it will remain at rest at that position for all times (True/False) **0.75 points**

This statement is true. As can be seen in the figure, $x = 2$ is a minimum of the potential and the particle will remain at rest at that location.

(b) If initially the particle is at rest at $x = 1$ then it will move towards the right as time increases, and as $t \to \infty$ then the position of the particle goes to $x \to \infty$ (True/False) **0.75 points**

This statement is false. The mechanical energy of the particle is E = V(1), there are two positions for which V(x) = E, as shown in the figure. Therefore the motion will be an oscillation between these two positions.

4. Consider a simple harmonic oscillator of mass $m$ and spring constant $k$. To this system we add an external force, $F_{ext}$, which is proportional to the velocity of the particle $\dot{x}$, with a constant of proporcionality $\beta > 0$, so that $F_{ext} = \beta\dot{x}$. In such a situation, the oscillations of the harmonic oscillator will be damped in the same way as in the damped harmonic oscillator (True/False) **1 point**

   This statement is false. The equation to be solved for this system is

   $$m\ddot{x} - \beta\dot{x} + kx = 0$$

   Looking for solutions of the form $x(t) = \exp(pt)$, we obtain the polynomial

   $$mp^2 - \beta p + k = 0.$$

   There is always at leat one root of the polynomial whose real part is positive, and therefore the oscillations of the harmonic oscillator are not damped. Instead they diverge to infinity.

5. The trajectory of a point particle which is in central force motion is a conic section as was postulated by Kepler (True/False) **1 point**

   This statement is false. This statement is true if the central force is proportional to $1/r^2$. Otherwise is false, in general.

6. Consider three points of a rigid body which are not collinear, $A$, $B$, and $C$. The body is in general motion (translation and rotation) with an angular velocity $\boldsymbol{\omega}$ such that $\boldsymbol{v_A} \cdot \boldsymbol{\omega} \neq 0$. If $|\boldsymbol{AB}| = |\boldsymbol{AC}|$ then the speed of $B$ is the same as the speed of $C$ (True/False) **1 point**

   This statement is false. The velocity of $B$ and $C$ can be expressed as

   $$\boldsymbol{v_B} = \boldsymbol{v_A} + \boldsymbol{\omega} \times \boldsymbol{AB},$$
   $$\boldsymbol{v_C} = \boldsymbol{v_A} + \boldsymbol{\omega} \times \boldsymbol{AC}.$$

   Both are the sum of two vectors. The fact that $|\boldsymbol{AB}| = |\boldsymbol{AC}|$, does not imply that $\boldsymbol{\omega} \times \boldsymbol{AB} = \boldsymbol{\omega} \times \boldsymbol{AC}$, and, in general, nothing can be said about the speed of $B$ and $C$.

7. Consider a homogeneous planar body of mass $m$ as shown in the figure, where the dimensions are also defined.

   (a) Determine the position of the centre of mass of the body, $G$, with respect to $Axy$ **1 point**.
   The position of the center of mass is given by

$$\frac{x_G}{h} = \frac{1}{2},$$
$$\frac{y_G}{h} = \frac{4 - 5t/h + 2t^2/h^2}{6 - 4t/h}.$$

   (b) Assuming that the moment of inertia tensor with respect to $Axy$, $\overline{\overline{I}}_A$, is known and given by

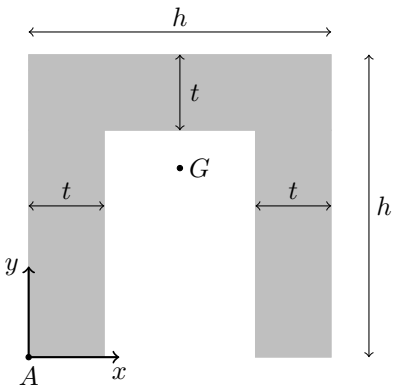
$$\overline{\overline{I}}_A = mh^2 \begin{bmatrix} \alpha & \beta & 0 \\ \beta & \gamma & 0 \\ 0 & 0 & \mu \end{bmatrix}$$

   where $\alpha$, $\beta$, $\gamma$ and $\mu$ are known constants, determine the moment of inertia tensor with respect to $Gxy$, $\overline{\overline{I}}_G$. **1 point**
   Using Steiner's theorem, we find

$$\overline{\overline{I}}_G = \overline{\overline{I}}_A - m \begin{bmatrix} y_G^2 & -x_G y_G & 0 \\ -x_G y_G & x_G^2 & 0 \\ 0 & 0 & x_G^2 + y_G^2 \end{bmatrix}.$$

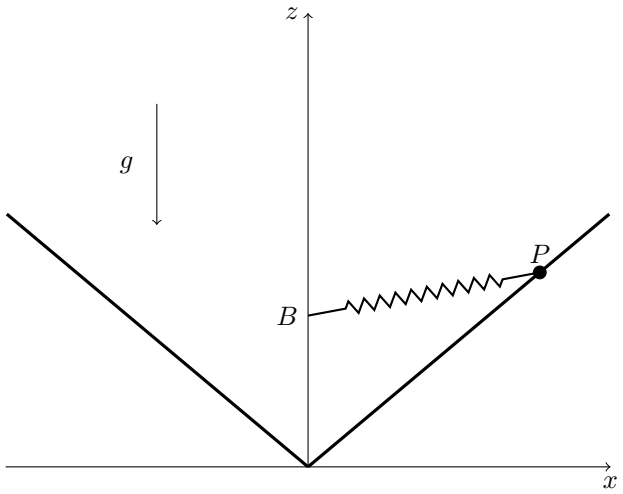
   (c) Without doing any integral, find a relation between $\beta$, $h$ and $t$. **1 point**
   The axes $Gxy$ are principal axes of inertia, therefore the product of inertia $P_{xy}$ must vanish. This yields the following relation

$$mh^2\beta + mx_G y_G = 0 \rightarrow \beta + \frac{4 - 5t/h + 2t^2/h^2}{12 - 8t/h} = 0.$$



# PROBLEM 1 *10 points, 60 minutes.*

A heavy point particle $P$ of mass $m$ is constrained to move on the surface of a smooth cone given in cylindrical coordinates by the equation $z = ar$ (in the region $z > 0$ only), where $a$ is a known constant. The particle is attached to a linear spring of constant $k$ with the other end attached to the point $B$ located on the $z-$axis at $z = z_0$.




1. Find the equilibrium positions. Recall that the motion is defined only for $z > 0$ so that $z = 0$ is not a valid equilibrium position. **2 points**

   The forces acting on the particle are the weight, $\boldsymbol{F_w}$, the spring force, $\boldsymbol{F_s}$ and the normal reaction, $\boldsymbol{N}$. A normal vector to the cone is given by $\boldsymbol{n} = -a\boldsymbol{e_r} + \boldsymbol{e_z}$. Then the three mentioned forces are

   $$\boldsymbol{F_w} = -mg\boldsymbol{e_z}, \tag{1}$$
   $$\boldsymbol{F_s} = -kr\boldsymbol{e_r} - k(z - z_0)\boldsymbol{e_z}, \tag{2}$$
   $$\boldsymbol{N} = -\lambda a\boldsymbol{e_r} + \lambda\boldsymbol{e_z}. \tag{3}$$

   In equilibrium, $\boldsymbol{F_w} + \boldsymbol{F_s} + \boldsymbol{N} = 0$. Solving this equation we obtain

   $$\frac{z_{eq}}{z_0} = \frac{1 - \dfrac{mg}{kz_0}}{1 + \dfrac{1}{a^2}}. \tag{4}$$

   Note that the whole local parallel at that height is an equilibrium position.

2. Compute the value of the normal reaction in the equilibrium positions. **1 point**

   The normal reaction is given by eq. (3) where the parameter $\lambda$ is

   $$\lambda = -\frac{kz_{eq}}{a^2}. \tag{5}$$

3. Discuss if the equilibrium positions can be located in the region $z > z_0/2$ as a function of the value of the ratio $mg/kz_0$, and for which values of $a$ is this solution physical. **1 point**

   The equilibrium position is above $z_0/2$ for

   $$\frac{mg}{kz_0} < \frac{1}{2}\left(1 - \frac{1}{a^2}\right). \tag{6}$$

   This solution is physical for $a > 1$, only.

Initially the particle is released at $z = z_0$ with a speed $v_0$ in the direction tangent to the local parallel.

4. Reduce the problem to quadratures. **3 points**

   We are looking for the functions $r(t)$, $\theta(t)$ and $z(t)$. A first relation is given by the equation of the cone $z = ar$. Note also that $\dot{z} = a\dot{r}$. A second relation is obtained by the observation that the forces acting on the particle do not have a component along the $\boldsymbol{e_\theta}$ direction. As a consequence the angular momentum with respect to the origin is conserved along the $\boldsymbol{e_z}$ direction. This leads to $r^2\dot{\theta} = C$, where $C$ can be obtained from the initial condition, leading to $C = v_0 z_0/a$. The third equation we need is the conservation of mechanical energy. The kinetic energy is given by

$$T = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2\right) = \frac{1}{2}m\left[(1+a^2)\dot{r}^2 + \frac{v_0^2 z_0^2}{a^2}\frac{1}{r^2}\right]. \tag{7}$$

   The potential energy is given by

$$V = mgz + \frac{1}{2}k[r^2 + (z - z_0)^2] = mgar + \frac{1}{2}k[r^2 + (ar - z_0)^2]. \tag{8}$$

   The mechanical energy is obtained imposing the initial conditions as

$$E = \frac{1}{2}mv_0^2 + mgz_0 + \frac{1}{2}k\frac{z_0^2}{a^2}. \tag{9}$$

   Thus, the equation $T + V = E$ is a relation of the form $f(r, \dot{r}) = 0$, and is the quadrature we are looking for. Then we can assume we know $r(t)$ and we obtain $z(t) = ar(t)$ and $\theta(t) = v_0 z_0/ar(t)^2$.

5. For the particular case $mg/kz_0 = 1/2$, $a = 1$, can you find a value of $v_0$ such that the particle remains always moving along the local parallel at $z = z_0$? **1 point**

   Imposing $a = 1$, the mechanical energy equation can be written as

$$m\dot{r}^2 = -\frac{1}{2}mv_0^2 z_0^2\frac{1}{r^2} - mgr - \frac{1}{2}k[r^2 + (r - z_0)^2] + \frac{1}{2}mv_0^2 + mgz_0 + \frac{1}{2}kz_0^2. \tag{10}$$

   We are looking for a solution in which $r = z_0$ at all times, so that $\dot{r} = 0$. We drop that term in the equation. We define $\xi = r/z_0$ and after some algebra, the equation to be solved is

$$-\frac{v_0^2}{2gz_0}\left(1 - \frac{1}{\xi^2}\right) + (\xi - 1) + \frac{kz_0}{mg}(\xi^2 - 1 + (\xi - 1)^2) = 0. \tag{11}$$

   The term $(\xi - 1)$ appears in all three terms of this equation. This means that $\xi = 1$ is already a solution of the equation. We simplify this term, and introduce the value given in the exercise $mg/kz_0 = 1/2$. This leads to the equation

$$-\frac{v_0^2}{2gz_0}\frac{\xi + 1}{\xi^2} + 1 + 2\xi = 0. \tag{12}$$

   $\xi = 1$ must be also a solution of this equation, otherwise the motion would not be confined to $r = z_0$. Instead, it would be an oscillation between two values of $r$, $r_{max}$ and $r_{min}$. Imposing that $\xi = 1$ is a solution we obtain that the initial velocity must be $v_0^2 = 3gz_0$.

   Note that the same result can be obtained in an easier way by using Newton's equations. Since the particle needs to stay at $z = z_0$, then $\dot{z} = \dot{r} = \ddot{z} = \ddot{r}$=0, and $r = z_0$ since $a = 1$. Then, the Newton's equations can be written

$$-mg - \lambda = 0, \tag{13}$$

$$-kz_0 - \lambda = -mz_0\dot{\theta}^2 = -m\frac{v_0^2}{z_0}. \tag{14}$$

   Imposing that $kz_0/mg = 2$ and solving for $\lambda$ and $v_0$ we obtain $\lambda = mg$ and $v_0^2 = 3gz_0$.

Finally, for the general case, consider that the surface of the cone is rough instead of smooth with a roughness coefficient $\mu$.

6. Write the down the equations of motion that you would need to solve in order to obtain the position of the particle as a function of time. **2 points**

   The friction force has a module $\mu|N| = \mu|\lambda\sqrt{1 + a^2}|$ and a direction which opposes the velocity of the particle. The velocity of the particle is given by

$$\boldsymbol{v} = \dot{r}\boldsymbol{e_r} + r\dot{\theta}\boldsymbol{e_\theta} + a\dot{r}\boldsymbol{e_z}, \tag{15}$$

so that the speed is $v = \sqrt{(1+a^2)\dot{r}^2 + r^2\dot{\theta}^2}$. Therefore the friction force can be written as

$$\boldsymbol{F_r} = -\frac{\mu|\lambda\sqrt{1+a^2}|}{\sqrt{(1+a^2)\dot{r}^2 + r^2\dot{\theta}^2}}(\dot{r}\boldsymbol{e_r} + r\dot{\theta}\boldsymbol{e_\theta} + a\dot{r}\boldsymbol{e_z}). \tag{16}$$

The acceleration of the point particle is given by

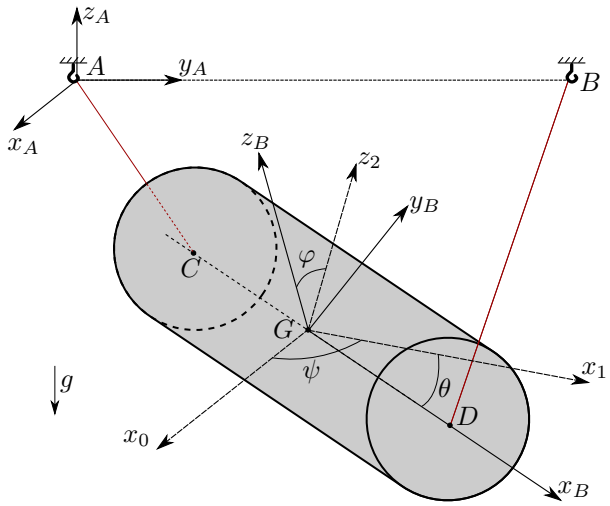
$$\boldsymbol{a} = (\ddot{r} - r\dot{\theta}^2)\boldsymbol{e_r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\boldsymbol{e_\theta} + a\ddot{r}\boldsymbol{e_z}. \tag{17}$$

The vector equation required is then $\boldsymbol{F_w} + \boldsymbol{F_s} + \boldsymbol{N} + \boldsymbol{F_r} = m\boldsymbol{a}$, where all the terms are given by equations (1), (2), (3), (16) and (17). This is a system of three equations with three unknowns, $r$, $\theta$ and $\lambda$.

## PROBLEM 2 *10 points, 90 minutes.*

A heavy, homogeneous cylinder of mass $m$, length $2l$, and radius $l/3$ is hanging from the ceiling from points $A$ and $B$, supported by two massless strings of length $l$ each. The distance between $A$ and $B$ is $3l$. The strings are always straight and connected to the cylinder at points $C$ and $D$ as shown in the figure. We call $G$ the center of mass of the cylinder. In order to describe the motion of the cylinder, we define the inertial reference frame $Ax_Ay_Az_A$ and the body-fixed reference frame $Gx_By_Bz_B$, as shown in the figure. We denote the position of $G$ in $Ax_Ay_Az_A$ as $\boldsymbol{r}_G = x_G\boldsymbol{i}_A + y_G\boldsymbol{j}_A + z_G\boldsymbol{k}_A$. The orientation of the cylinder is described by the Euler angles $\psi$, $\theta$, $\varphi$, where: $\psi$ is a simple rotation along $\boldsymbol{k}_0$ from $Gx_0y_0z_0$ (which are parallel to $Ax_Ay_Az_A$ ) to $Gx_1y_1z_1$; $\theta$ is a simple rotation along $\boldsymbol{j}_1$ from $Gx_1y_1z_1$ to $Gx_2y_2z_2$; and $\varphi$ is a simple rotation along $\boldsymbol{i}_2$ from $Gx_2y_2z_2$ to $Gx_By_Bz_B$. Observe that for clarity the figure only displays part of the auxiliary axes of these rotations. Lastly, the tensor of inertia of the cylinder in $Gx_By_Bz_B$ is given in terms of two known constants $\lambda$ and $\mu$,

$$\bar{\bar{I}} = ml^2 \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \mu \end{bmatrix}$$




1. Write down the position vectors of $C$ and $D$ in $Ax_Ay_Az_A$ as a function of the variables of the problem. **1.5 points**

   We first define the vector bases of all the reference frames used:

   $$\begin{cases} \boldsymbol{i}_1 = \cos\psi\boldsymbol{i}_A + \sin\psi\boldsymbol{j}_A \\ \boldsymbol{j}_1 = -\sin\psi\boldsymbol{i}_A + \cos\psi\boldsymbol{j}_A \\ \boldsymbol{k}_1 = \boldsymbol{k}_A \end{cases} \quad \begin{cases} \boldsymbol{i}_2 = \cos\theta\boldsymbol{i}_1 - \sin\theta\boldsymbol{k}_1 \\ \boldsymbol{j}_2 = \boldsymbol{j}_1 \\ \boldsymbol{k}_2 = \sin\theta\boldsymbol{i}_1 + \cos\theta\boldsymbol{k}_1 \end{cases} \quad \begin{cases} \boldsymbol{i}_B = \boldsymbol{i}_2 \\ \boldsymbol{j}_B = \cos\varphi\boldsymbol{j}_2 + \sin\varphi\boldsymbol{k}_2 \\ \boldsymbol{k}_B = -\sin\varphi\boldsymbol{j}_2 + \cos\varphi\boldsymbol{k}_2 \end{cases}$$

   Then, constructing the position vectors of $C$, $D$ is straightforward:

   $$\boldsymbol{r}_C = \boldsymbol{r}_G - l\boldsymbol{i}_B = (x_G - l\cos\theta\cos\psi)\,\boldsymbol{i}_A + (y_G - l\cos\theta\sin\psi)\,\boldsymbol{j}_A + (z_G + l\sin\theta)\,\boldsymbol{k}_A$$
   $$\boldsymbol{r}_D = \boldsymbol{r}_G + l\boldsymbol{i}_B = (x_G + l\cos\theta\cos\psi)\,\boldsymbol{i}_A + (y_G + l\cos\theta\sin\psi)\,\boldsymbol{j}_A + (z_G - l\sin\theta)\,\boldsymbol{k}_A$$

2. Calculate the angular velocity and angular acceleration of the body as a function of the variables of the problem and their derivatives. **1.5 points**

   $$\begin{aligned} \boldsymbol{\omega}_{B/A} &= \boldsymbol{\omega}_{1/A} + \boldsymbol{\omega}_{2/1} + \boldsymbol{\omega}_{B/2} = \dot{\psi}\boldsymbol{k}_A + \dot{\theta}\boldsymbol{j}_1 + \dot{\varphi}\boldsymbol{i}_B \\ &= -\left(\dot{\psi}\sin\theta - \dot{\varphi}\right)\boldsymbol{i}_B + \left(\dot{\psi}\cos\theta\sin\varphi + \dot{\theta}\cos\varphi\right)\boldsymbol{j}_B + \left(\dot{\psi}\cos\theta\cos\varphi - \dot{\theta}\sin\varphi\right)\boldsymbol{k}_B \end{aligned}$$

   Taking derivatives on $\boldsymbol{\omega}_{B/A}$ (remember that the derivatives of $\boldsymbol{\omega}_{B/A}$ in $Ax_Ay_Az_A$ and in $Gx_By_Bz_B$ yield the same physical vector):

   $$\begin{aligned} \boldsymbol{\alpha}_{B/A} = &-\left(\ddot{\psi}\sin\theta + \dot{\psi}\dot{\theta}\cos\theta - \ddot{\varphi}\right)\boldsymbol{i}_B \\ &+ \left(\ddot{\psi}\cos\theta\sin\varphi - \dot{\psi}\dot{\theta}\sin\theta\sin\varphi + \dot{\psi}\dot{\varphi}\cos\theta\cos\varphi + \ddot{\theta}\cos\varphi - \dot{\theta}\dot{\varphi}\sin\varphi\right)\boldsymbol{j}_B \\ &+ \left(\ddot{\psi}\cos\theta\cos\varphi - \dot{\psi}\dot{\theta}\sin\theta\cos\varphi - \dot{\psi}\dot{\varphi}\cos\theta\sin\varphi - \ddot{\theta}\sin\varphi - \dot{\theta}\dot{\varphi}\cos\varphi\right)\boldsymbol{k}_B \end{aligned}$$

(Note: results could be given in other axes. Working in $Gx_By_Bz_B$ is advisable as it will simplify writing the equations of Euler. Working in $Gx_2y_2z_2$ is also possible and results in somewhat shorter expressions).

3. Write down the equation of the constraints imposed by the strings. **1.5 points**

   The two constraints are $|\boldsymbol{r}_C - \boldsymbol{r}_A| = l$ and $|\boldsymbol{r}_D - \boldsymbol{r}_B| = l$, i.e., points $C$ and $D$ move on spheres of radius $l$ centered on $A$ and $B$, respectively. Expanding these expressions we obtain:

$$(x_G - l\cos\theta\cos\psi)^2 + (y_G - l\cos\theta\sin\psi)^2 + (z_G + l\sin\theta)^2 = l^2 \tag{18}$$

$$(x_G + l\cos\theta\cos\psi)^2 + (y_G + l\cos\theta\sin\psi - 3l)^2 + (z_G - l\sin\theta)^2 = l^2 \tag{19}$$

4. Write down the expressions of all forces and moments of forces at $G$ acting on the cylinder. **2.5 points**

   There are three forces acting on the cylinder: the tension of the $\boldsymbol{AC}$ string, $\boldsymbol{T}_1$, acting on $C$, the tension of the $\boldsymbol{BD}$ string, $\boldsymbol{T}_2$, acting on $D$, and the weight, $\boldsymbol{W}$ acting on $G$:

$$\begin{aligned}
\boldsymbol{T}_1 &= -T_1\left(x_G/l - \cos\theta\cos\psi\right)\boldsymbol{i}_A - T_1\left(y_G/l - \cos\theta\sin\psi\right)\boldsymbol{j}_A - T_1\left(z_G/l + \sin\theta\right)\boldsymbol{k}_A\\
\boldsymbol{T}_2 &= -T_2\left(x_G/l + \cos\theta\cos\psi\right)\boldsymbol{i}_A - T_2\left(y_G/l + \cos\theta\sin\psi - 3\right)\boldsymbol{j}_A - T_2\left(z_G/l - \sin\theta\right)\boldsymbol{k}_A\\
\boldsymbol{W} &= -mg\boldsymbol{k}_A
\end{aligned}$$

   where $T_1$ and $T_2$ are two unknowns to be determined as part of the solution.

   The forces $\boldsymbol{T}_1$ and $\boldsymbol{T}_2$ create a moment on $G$. We can take into account that $\boldsymbol{GD} = -\boldsymbol{GC} = l\boldsymbol{i}_B = l\cos\theta\cos\psi\boldsymbol{i}_A + l\cos\theta\sin\psi\boldsymbol{j}_A - l\sin\theta\boldsymbol{k}_A$, to partially simplify the cross products with $\boldsymbol{T}_1$ and $\boldsymbol{T}_2$, obtaining the following results:

$$\begin{aligned}
\boldsymbol{\mathcal{M}}^G_{T_1} = \boldsymbol{GC}\times\boldsymbol{T}_1 &= T_1\left(z_G\cos\theta\sin\psi + y_G\sin\theta\right)\boldsymbol{i}_A - T_1\left(x_G\sin\theta + z_G\cos\theta\cos\psi\right)\boldsymbol{j}_A\\
&+ T_1\left(y_G\cos\theta\cos\psi - x_G\cos\theta\sin\psi\right)\boldsymbol{k}_A
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{\mathcal{M}}^G_{T_2} = \boldsymbol{GD}\times\boldsymbol{T}_2 &= -T_2\left(z_G\cos\theta\sin\psi + (y_G - 3)\sin\theta\right)\boldsymbol{i}_A + T_2\left(x_G\sin\theta + z_G\cos\theta\cos\psi\right)\boldsymbol{j}_A\\
&- T_2\left((y_G - 3)\cos\theta\cos\psi - x_G\cos\theta\sin\psi\right)\boldsymbol{k}_A
\end{aligned}$$

5. What is the total number of *unknowns* in the problem? How many equations are needed to solve the problem? **0.5 points**

   There are 6 variables and 2 unknown reaction forces, so the total number of unknowns is 8. We need 8 equations. The first two are the equations of the constraints derived above. The other 6 are the equations of motion of the cylinder (see next question).

6. Write down the equations of motion of the cylinder (plus any other necessary equations) that if integrated, would yield the evolution in time of these unknowns. Note that you do not need to integrate them. **1 point**

   As stated above, to the following 6 equations we need to add the 2 constraints of previous questions:

$$\begin{aligned}
m\ddot{x}_G &= -\left(T_1 + T_2\right)x_G/l + \left(T_1 - T_2\right)\cos\theta\cos\psi\\
m\ddot{y}_G &= -\left(T_1 + T_2\right)y_G/l + \left(T_1 - T_2\right)\cos\theta\sin\psi + 3T_2\\
m\ddot{z}_G &= -\left(T_1 + T_2\right)z_G/l - \left(T_1 - T_2\right)\sin\theta - mg
\end{aligned}$$

   For the rotational equations, we take the derivatives in the $Gx_By_Bz_B$ reference frame applying Coriolis transport theorem:

$$\begin{aligned}
ml^2\lambda\alpha_{x_B} &= 0 \quad \Rightarrow \dot{\psi}\sin\theta - \dot{\varphi} = \text{const}\\
ml^2\mu\alpha_{y_B} - (\mu - \lambda)\,\omega_{x_B}\omega_{z_B} &= -\left(T_1 - T_2\right)\left[x_G\left(\cos\psi\sin\theta\cos\varphi + \sin\psi\sin\varphi\right)\right.\\
&\quad + y_G\left(\sin\psi\sin\theta\cos\varphi - \cos\psi\sin\varphi\right) + z_G\cos\theta\cos\varphi\left.\right]\\
&\quad - 3T_2\left(\sin\psi\sin\theta\cos\varphi - \cos\psi\sin\varphi\right)\\
ml^2\mu\alpha_{z_B} + (\mu - \lambda)\,\omega_{x_B}\omega_{y_B} &= \left(T_1 - T_2\right)\left[x_G\left(\cos\psi\sin\theta\sin\varphi - \sin\psi\cos\varphi\right)\right.\\
&\quad + y_G\left(\sin\psi\sin\theta\sin\varphi + \cos\psi\sin\varphi\right) + z_G\cos\theta\cos\varphi\left.\right]\\
&\quad + 3T_2\left(\sin\psi\sin\theta\sin\varphi + \cos\psi\sin\varphi\right)
\end{aligned}$$

7. Is the mechanical energy of the cylinder conserved in this problem? If so, write down its equation of conservation of energy. Is this an independent equation with respect to the set of equations of the previous question? **1.5 points**

   Yes, it is conserved, as the tension forces are always normal to the velocity of the application point for each string (the points move on the surface of a sphere, while the tension is directed toward the center of that sphere), so they effect no work. Clearly, this is not an independent equation since we already have a closed system of equations for the cylinder. It can, however, be used to substitute one of the existing equations.

$$T = \frac{1}{2}m\left(\dot{x}_G^2 + \dot{y}_G^2 + \dot{z}_G^2\right) + \frac{1}{2}ml^2\left(\lambda\omega_{x_B}^2 + \mu\omega_{y_B}^2 + \mu\omega_{z_B}^2\right)$$
$$V = mgz_G$$

   The conservation of energy equation is simply $T + V = \text{const}$.