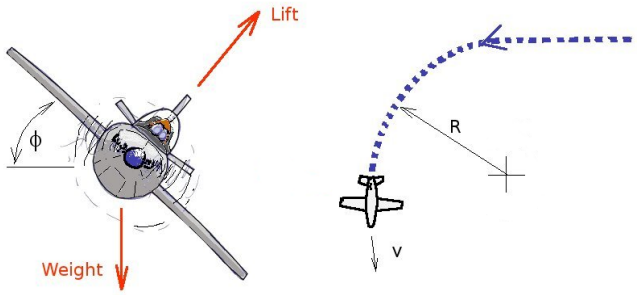
THEORY

1/3 of the grade, 60 minutes.

Please answer the questions 1–8. Those questions requiring a True or False answer must have a short explanation of your answer. **A right answer without explanation will not be graded**.

1. A heavy aircraft of mass m is performing a level turn at constant speed v and constant radius of curvature R. Find the value of the angle of bank ϕ as a function of the parameters of the problem, and the load factor $n = L/W$ as a function of ϕ. **(1 point)**.



Since

$$L\cos\phi = mg \tag{1}$$

$$L\sin\phi = mv^2/R \tag{2}$$

we find

$$\tan\phi = \frac{v^2}{gR}; \quad n = \frac{1}{\cos\phi} \tag{3}$$

2. The motion of a particle in a plane has constant speed and describes a spiral given by the following parametric equation in polar coordinates:

$$r = 2\theta. \tag{4}$$

The tangential acceleration of the particle is non-zero (True/False). **(1 point)**.

False. By definition, the tangential acceleration gives the time variation of the speed, and in this case the speed is constant, so the tangential acceleration is zero.

3. A vector $\boldsymbol{v}$ is constant when seen from a reference frame $O'X'Y'Z'$ which is rotating and translating with respect to another reference frame $OXYZ$. The time derivative of $\boldsymbol{v}$ as seen from $OXYZ$ can have a component parallel to $\boldsymbol{v}$ itself (True/False). **(1 point)**.

False. Using Coriolis theorem, $d\boldsymbol{v}/dt|_1 = d\boldsymbol{v}/dt|_0 + \boldsymbol{\omega}_{01} \times \boldsymbol{v}$. Since the derivative in $O'X'Y'Z'$ is zero, the derivative in $OXYZ$ must always be perpendicular to $\boldsymbol{v}$ itself.

4. A particle of mass m is moving in a potential given by $V(x) = \sin(x)/x$. Indicate in a reasoned way which of the following statements are true:

   (a) If the particle is released at $x = \pi$ with zero velocity, the particle will remain at rest at that position indefinitely (True/False). **(0.75 points)**.
   False: At that point $dV/dx \neq 0$, so there is a force acting on the particle that will set it in motion.

   (b) If we release the particle at any point $x > 0$ with a mechanical energy $E = 1$ and an initial velocity toward the right, it will move toward $x \to \infty$ as $t \to \infty$, and the speed it will have at infinity is $v = \sqrt{2/m}$ (True/False). **(0.75 points)**.
   True: the potential is always $V < 1$ for $x > 0$. Therefore, the particle does not find any potential barrier it cannot overcome in its motion toward the right. At $x \to \infty$, $V \to 0$, and so all the mechanical energy is kinetic energy. This means that $v = \sqrt{2E/m}$.

5. A particle of mass m is held to the floor and to the ceiling of the room with two springs of stiffness k each. Indicate in a reasoned way which of the following statements are true:

   (a) If there is no damping, the natural frequency of the system is $\omega_0 = \sqrt{2k/m}$ (True/False). **(0.75 points)**.

   True: If we call $z = z_0 + z'$ the distance from the floor, with z_0 the position of equilibrium of the particle and z' the displacement from it, the force that appears on the particle due to the two springs combined when $z' \neq 0$ is $F_z = -2kz'$. Therefore, the two springs together are equivalent to a single spring of stiffness $2k$, in all that regards the motion of the particle with respect to its position of equilibrium. The natural frequency of the system is therefore $\omega_0 = \sqrt{2k/m}$ (by definition, or by solving the equation of the harmonic oscillator).

   (b) If an external force F_1 pushes the particle up and down periodically with a very high frequency $\omega_1 \gg \omega_0$, the particle will go up as the force is pushing down, and will go down as the force is pushing up (True/False). Assume in this case that there is a small non-zero damping due to air drag. **(0.75 points)**.

   Hint: remember that the amplitude a_1 and phase difference θ_1 of a forced harmonic oscillator with damping coefficient γ are given by:

$$a_1 = \frac{F_1/m}{\sqrt{(\omega_0^2 - \omega_1^2)^2 + 4\gamma^2\omega_1^2}}; \quad \tan\theta_1 = \frac{2\gamma\omega_1}{\omega_0^2 - \omega_1^2} \tag{5}$$

   True: At high forcing frequency, the particle and the force are completely out of phase: The phase difference between the force and the motion of the particle is given by $\tan\theta_1 = 2\gamma\omega_1/(\omega_0^2 - \omega_1^2)$. As $\omega_1 \gg \omega_0$, $\tan\theta_1 \simeq 0^-$, so $\theta_1 \simeq \pi$.

6. The international space station is orbiting around the Earth when a resupply vehicle rendezvouses and docks with it, increasing the mass of the orbiting body in a 10%. As a consequence, its orbital period will be reduced in a 10% (True/False). **(1 point)**.

   False: According to Keplers third law, the period of a satellite depends on the gravitational parameter of the central body and the semimajor axis of the ellipse, which are both independent of the satellite mass.

7. A rigid body is under general motion (rotation and translation). Indicate in a reasoned way which of the following statements are true:

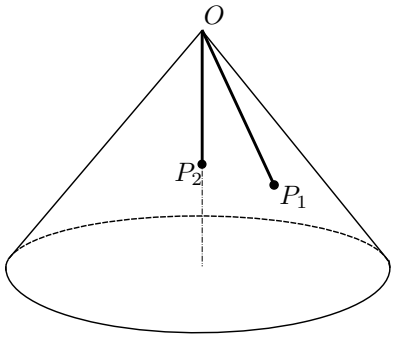
   (a) Given the velocity vectors $\boldsymbol{v}_A$, $\boldsymbol{v}_B$, $\boldsymbol{v}_C$ of any three known points A, B, C of the body it is always possible to compute its angular velocity vector, $\boldsymbol{\omega}$ (True/False). **(0.75 points)**.

   False: if the points are co-linear, it is not possible to compute all components of $\boldsymbol{\omega}$.

   (b) Given the angular velocity vector $\boldsymbol{\omega}$ and the acceleration vectors $\boldsymbol{a}_D$, $\boldsymbol{a}_E$, $\boldsymbol{a}_F$ of any three known points D, E, F of the body that define a plane, it is always possible to compute its angular acceleration vector, $\boldsymbol{\alpha}$ (True/False). **(0.75 points)**.

   True: we can write $\boldsymbol{a}_E - \boldsymbol{a}_D - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{DE}) = \boldsymbol{\alpha} \times \boldsymbol{DE}$ and $\boldsymbol{a}_F - \boldsymbol{a}_D - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{DF}) = \boldsymbol{\alpha} \times \boldsymbol{DF}$, where all vectors in the left hand side of the equations are known. Since the three points are not co-linear (they define a plane), the two vector equations allow to find the three components of $\boldsymbol{\alpha}$ (to do it, it is convenient to cross-multiply them by $\boldsymbol{\alpha}$ first).

8. As shown in the figure below, two point particles P_1 and P_2 are connected by a string of constant length ℓ. Particle P_1 is constrained to move without friction on the surface of a hollow cone (the particle cannot leave the surface). The string passes through an aperture at the cone vertex O, and it can slide through it without friction. Particle P_2 hangs along the cone axis, and is constrained to move up and down along it. The string is always tense and the particles do not reach point O in their motion. How many degrees of freedom has the problem? define a minimum set of valid coordinates to characterize those degrees of freedom. **(1.5 points)**.

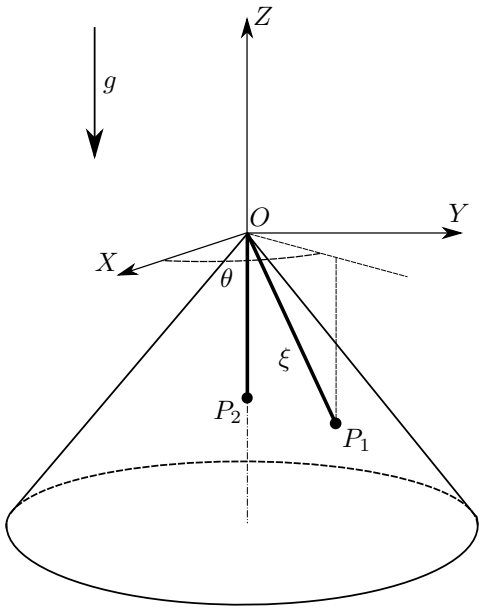


The system has two degrees of freedom: P_1 can move along a parallel or along a meridian of the cone. Note that P_2 does not add any extra degree of freedom, as its up-down motion is completely dictated by the motion of P_1 along the meridians of the cone. A valid set of two coordinates to study the problem are ξ, the length of string from O to P_1, and θ, the angle between the OP_1P_2 plane and any fixed reference plane that contains the axis of the cylinder.

PROBLEM 1

1/3 of the grade, 75 minutes.

Two heavy point particles each of mass m are connected by an inextensible, massless string of total length ℓ. Particle P_1 is constrained to move without friction over the surface of a cone of semiangle α (the particle cannot leave the surface of the cone). The string passes through an aperture at the cone vertex O, and it can slide through it without friction. Particle P_2 hangs along the cone axis, and is constrained to move along it. We define a set of inertial axes $OXYZ$ as shown in the figure. We call ξ the length of the fraction of string between O and P_1, and θ the angle formed between the X axis and the projection onto the XY plane of the OP_1 vector. In the following, assume that the string is always tense and that the particles do not reach point O during motion.



1. Write down the general expression of the position, velocity and acceleration vectors of P_1 and P_2 as a function of θ, ξ and the parameters of the problem **(3 points)**.

   Observe that working in cylindrical coordinates ($\boldsymbol{e}_r = \cos\theta\boldsymbol{i} + \sin\theta\boldsymbol{j}$ and $\boldsymbol{e}_\theta = -\sin\theta\boldsymbol{i} + \cos\theta\boldsymbol{j}$) simplifies calculations:

$$\boldsymbol{r}_1 = \xi\sin\alpha\boldsymbol{e}_r - \xi\cos\alpha\boldsymbol{k} = \xi\sin\alpha(\cos\theta\boldsymbol{i} + \sin\theta\boldsymbol{j}) - \xi\cos\alpha\boldsymbol{k} \tag{6}$$

$$\boldsymbol{v}_1 = \dot{\xi}\sin\alpha\boldsymbol{e}_r + \xi\dot{\theta}\sin\alpha\boldsymbol{e}_\theta - \dot{\xi}\cos\alpha\boldsymbol{k} = \tag{7}$$

$$= \sin\alpha(\dot{\xi}\cos\theta - \xi\dot{\theta}\sin\theta)\boldsymbol{i} + \sin\alpha(\dot{\xi}\sin\theta + \xi\dot{\theta}\cos\theta)\boldsymbol{j} - \dot{\xi}\cos\alpha\boldsymbol{k} \tag{8}$$

$$\boldsymbol{a}_1 = (\ddot{\xi} - \xi\dot{\theta}^2)\sin\alpha\boldsymbol{e}_r + (2\dot{\xi}\dot{\theta} + \xi\ddot{\theta})\sin\alpha\boldsymbol{e}_\theta - \ddot{\xi}\cos\alpha\boldsymbol{k} = \tag{9}$$

$$= \sin\alpha[(\ddot{\xi} - \xi\dot{\theta}^2)\cos\theta - (2\dot{\xi}\dot{\theta} + \xi\ddot{\theta})\sin\theta]\boldsymbol{i} + \sin\alpha[(\ddot{\xi} - \xi\dot{\theta}^2)\sin\theta + (2\dot{\xi}\dot{\theta} + \xi\ddot{\theta})\cos\theta]\boldsymbol{j} - \ddot{\xi}\cos\alpha\boldsymbol{k} \tag{10}$$

   For P_2, the solution is simply:

$$\boldsymbol{r}_2 = -(\ell - \xi)\boldsymbol{k}; \quad \boldsymbol{v}_2 = \dot{\xi}\boldsymbol{k}; \quad \boldsymbol{a}_2 = \ddot{\xi}\boldsymbol{k}. \tag{11}$$

2. Write down the expression of all forces that act on P_1 and P_2 **(1 point)**.

   The forces on P_1 are the weight $\boldsymbol{W}_1 = -mg\boldsymbol{k}$, the tension $\boldsymbol{T}_1 = T(-\sin\alpha\boldsymbol{e}_r + \cos\alpha\boldsymbol{k})$, and the normal reaction of the cone $\boldsymbol{N} = N(\cos\alpha\boldsymbol{e}_r + \sin\alpha\boldsymbol{k})$.

The forces on P_2 are the weight $\boldsymbol{W}_2 = -mg\boldsymbol{k}$ and the tension $\boldsymbol{T}_2 = T\boldsymbol{k}$.

Note that T and N are two additional unknowns in the problem, but that there is a single value of T for the two particles.

3. Write down the differential equations that, if solved, would yield the time evolution of θ, ξ, the tension on the string and any other reaction force in the problem **(1 point)**.

   Newton's equations for particle P_1 read:

$$m(\ddot{\xi} - \xi\dot{\theta}^2)\sin\alpha = -T\sin\alpha + N\cos\alpha \tag{12}$$

$$m(2\dot{\xi}\dot{\theta} + \xi\ddot{\theta})\sin\alpha = 0 \tag{13}$$

$$-m\ddot{\xi}\cos\alpha = -mg + T\cos\alpha + N\sin\alpha \tag{14}$$

   and for P_2:

$$m\ddot{\xi} = -mg + T \tag{15}$$

4. What is the mathematical condition that the function $\xi(t)$ must satisfy for the string to remain tense all the time? **(1 point)**.

   From the last equation above, it is required that $\ddot{\xi} > -g$.

5. How fast should P_1 rotate in the θ direction to have a circular trajectory over a cone parallel? What is the value of the string tension and any other reaction force in the problem in that case? Give your answer as a function of ξ and the parameters of the problem **(2 points)**.

   Using the equations above to eliminate N and T, and taking $\dot{\xi}, \ddot{\xi}, \ddot{\theta} = 0$ one finds the required condition is:

$$\xi\dot{\theta}^2\sin^2\alpha = g(1-\cos\alpha). \tag{16}$$

   From the other equations, $T = mg$ and $N = mg(1-\cos\alpha)/\sin\alpha$

6. Taking these equations as the starting point (or based on the physical considerations of the problem) write down two first-integrals of the problem that do not depend on the string tension or any other reaction force in the problem **(2 points)**.

   The two equations are the conservation of angular momentum of P_1 with respect to the Z axis (the force problem on P_1 is a central force problem, as there is no force terms along $\boldsymbol{e}_\theta$) and the conservation of mechanical energy of the system (all forces other than tension are conservative or do not work; the tension works identically on each of the two particles, removing energy from one and adding it exactly to the other). From this it is direct to reach the expressions below without any derivations.

   Alternatively, one can work with the equations above to obtain these results. The first one is the integral of the second equation above. After multiplying by $\xi\sin\alpha$, the equation becomes integrable into

$$m\xi^2\dot{\theta}\sin^2\alpha = H_{Z1}. \tag{17}$$

   The second one can be found by cancelling T and N from the equations above and multiplying by $\dot{\xi}$. The resulting equation becomes integrable into the following (one can use the definition of H_{Z1} to perform the integration):

$$m\dot{\xi}^2 + \frac{1}{2}m\xi^2\dot{\theta}^2\sin^2\alpha + mg\xi(1-\cos\alpha) = E. \tag{18}$$

PROBLEM 2

1/3 of the grade, 75 minutes.

In this problem, we are going to study the motion of a heavy, axisymmetric pegtop of mass m. We will assume that during the precession, nutation and spin of the pegtop, the vertex O of the pegtop remains fixed at the origin of the inertial reference frame $Oxyz$. The distance ℓ between O and the center of mass of the pegtop G is known. Also the moment of inertia tensor with respect to the principal axes of inertia $Gx_By_Bz_B$ is known, and given by

$$\overline{\overline{I}}_G = m\ell^2 \begin{bmatrix} A & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & B \end{bmatrix}.$$

The orientation of the body fixed reference frame $Gx_By_Bz_B$ with respect to the inertial reference frame $Oxyz$ is given by three sequential rotations. First, a rotation ψ with respect to the axis Oz, (leading to an intermediate reference frame $Ox_1y_1z_1$). Second, a rotation θ with respect to the axis Ox_1, (leading to an intermediate reference frame $Ox_2y_2z_2$). And finally a rotation φ with respect to the axis Gz_2, leading to $Gx_By_Bz_B$. The initial condition is $\psi = 0$, $\theta = \pi/6$, $\varphi = 0$, $\dot{\psi} = 0$, $\dot{\theta} = \Omega_1$, $\dot{\varphi} = \Omega_2$, where Ω_1 and Ω_2 are known constants.

1. Compute the inertia tensor, $\overline{\overline{I}}_O$ with respect to a reference frame, parallel to $Gx_By_Bz_B$ with origin in O, i.e. $Ox_By_Bz_B$. **(1 point)**.

   Since $\boldsymbol{r}_G = \ell \boldsymbol{k_B}$, using Steiner's theorem we obtain

   $$\overline{\overline{I}}_O = m\ell^2 \begin{bmatrix} A+1 & 0 & 0 \\ 0 & A+1 & 0 \\ 0 & 0 & B \end{bmatrix}.$$

2. Compute the angular momentum of the peg-top with respect to O as a function of the variables of the problem at an arbitrary instant of time. **(1 point)**.

   Since the peg-top is axisymmetric, it is convenient to use the reference frame $0x_2y_2z_2$ although it is not a body-fixed reference frame. The angular velocity of the body with respect to $Oxyz$ is given by

   $$\boldsymbol{\omega}_B = \dot{\theta}\boldsymbol{i_2} + \dot{\psi}\sin\theta \boldsymbol{j_2} + (\dot{\varphi} + \dot{\psi}\cos\theta)\boldsymbol{k_2}, \tag{19}$$

   while the angular velocity of the reference frame $Ox_2y_2z_2$ with respect to $Oxyz$ is given by

   $$\boldsymbol{\omega}_B = \dot{\theta}\boldsymbol{i_2} + \dot{\psi}\sin\theta \boldsymbol{j_2} + \dot{\psi}\cos\theta \boldsymbol{k_2}. \tag{20}$$

   Since O is fixed the angular momentum of the body with respect to O is given by $\boldsymbol{H_O} = \overline{\overline{I}}_O \boldsymbol{\omega}_B$,

   $$\boldsymbol{H}_O = m\ell^2 \left[(A+1)\dot{\theta}\boldsymbol{i_2} + (A+1)\dot{\psi}\sin\theta \boldsymbol{j_2} + B(\dot{\varphi} + \dot{\psi}\cos\theta)\boldsymbol{k_2} \right]. \tag{21}$$

3. Compute the kinetic energy of the peg-top as a function of the variables of the problem. **(1 point)**.

   The kinetic energy is given by

   $$T = \frac{1}{2}\boldsymbol{\omega}_B \overline{\overline{I}}_O \boldsymbol{\omega}_B = \frac{1}{2} m\ell^2 \left[(A+1)(\dot{\theta}^2 + \dot{\psi}^2\sin^2\theta) + B(\dot{\varphi} + \dot{\psi}\cos\theta)^2 \right]. \tag{22}$$

4. Is the mechanical energy of the system conserved? Why? **(1 point)**.

   The external forces acting on the body are the weight which is conservative and the normal reaction at O. The latter does not perform work because O is a fixed point. Therefore the mechanical energy is conserved.

5. Is there any conserved component of the angular momentum with respect to O? Why? **(1 point)**.

   The torque acting on O is given by $\boldsymbol{M}_O = \boldsymbol{r_G} \times (-mg\boldsymbol{k})$. Due to the properties of the vector product, $\boldsymbol{M}_O \cdot \boldsymbol{k} = 0$. As a consequence the vertical component of the angular momentum with respect to O is conserved $\boldsymbol{H}_O \cdot \boldsymbol{k}$ =constant, since $\mathrm{d}\boldsymbol{H}_O/\mathrm{d}t = \boldsymbol{M}_O$.

6. Obtain three **first order** ordinary differential equations that would allow to obtain the values of ψ, θ and φ as a function of time. **(4 points)**.

   The first equation is the conservation of mechanical energy. We have already computed the kinetic energy, we need to compute the potential energy. This is given by $V = mgz_G$. In order to express z_G as a function of the variables of the problem, we recall that $\boldsymbol{r_G} = \ell \boldsymbol{k_B}$, so that

$$z_G = \boldsymbol{r_G} \cdot \boldsymbol{k} = \ell \boldsymbol{k_B} \cdot \boldsymbol{k} = \ell \cos\theta,$$
$$V = mg\ell \cos\theta.$$

   Then, the conservation of mechanical energy is

$$\frac{1}{2} m\ell^2 \left[(A+1)(\dot{\theta}^2 + \dot{\psi}^2 \sin^2\theta) + B(\dot{\varphi} + \dot{\psi}\cos\theta)^2 \right] + mg\ell\cos\theta = E, \tag{23}$$

   where E is obtained from the initial conditions

$$E = \frac{1}{2} m\ell^2 \left[(A+1)\Omega_1^2 + B\Omega_2^2 \right] + \frac{\sqrt{3}}{2} mg\ell. \tag{24}$$

   The second equation is obtained from the conservation of angular momentum along the vertical direction $\boldsymbol{H}_O \cdot \boldsymbol{k}$ =constant. Note that $\boldsymbol{k} = \boldsymbol{k_1} = \cos\theta \boldsymbol{k_2} + \sin\theta \boldsymbol{j_2}$. The scalar product yields

$$(A+1)\dot{\psi}\sin^2\theta + B\dot{\varphi}\cos\theta + B\dot{\psi}\cos^2\theta = \frac{\sqrt{3}}{2} B\Omega_2. \tag{25}$$

   Finally, in order to obtain the third equation, we need to consider the moments equation (Euler equation)

$$\boldsymbol{M_O} = \left[\frac{\mathrm{d}\boldsymbol{H_O}}{\mathrm{d}t} \right]_{Ox_2y_2z_2} + \boldsymbol{\omega_2} \times \boldsymbol{H_O}. \tag{26}$$

   First,

$$\boldsymbol{M_O} = \boldsymbol{r_G} \times (-mg\boldsymbol{k}) = -mg\ell[\boldsymbol{k_2} \times (\cos\theta \boldsymbol{k_2} + \sin\theta \boldsymbol{j_2})] = mg\ell \sin\theta \boldsymbol{i_2}. \tag{27}$$

   In addition, the component along $\boldsymbol{k_2}$ of the vector product $\boldsymbol{\omega_2} \times \boldsymbol{H_O}$ is zero. Therefore, the third equation we are looking for is the third Euler equation

$$\frac{\mathrm{d}}{\mathrm{d}t}(\dot{\varphi} + \dot{\psi}\cos\theta) = 0, \tag{28}$$

   and integrating this equation and imposing the initial condition, we obtain

$$\dot{\varphi} + \dot{\psi}\cos\theta = \Omega_2 \tag{29}$$

   In summary, the three equations requested are

$$(A+1)(\dot{\theta}^2 + \dot{\psi}^2\sin^2\theta) + B(\dot{\varphi} + \dot{\psi}\cos\theta)^2 + \frac{2g}{\ell}\cos\theta = (A+1)\Omega_1^2 + B\Omega_2^2 + \frac{\sqrt{3}g}{\ell}, \tag{30}$$

$$(A+1)\dot{\psi}\sin^2\theta + B\dot{\varphi}\cos\theta + B\dot{\psi}\cos^2\theta = \frac{\sqrt{3}}{2} B\Omega_2, \tag{31}$$

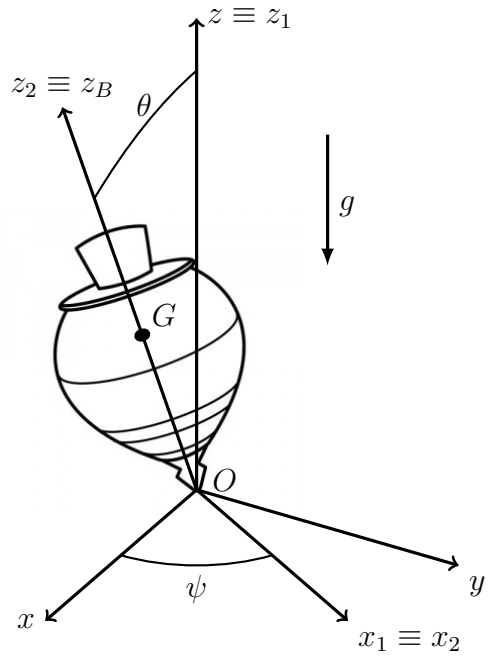
$$\dot{\varphi} + \dot{\psi}\cos\theta = \Omega_2. \tag{32}$$

7. Assuming that $\psi(t)$, $\theta(t)$ and $\varphi(t)$ are known, explain how you would obtain the value of the reaction at O. You do not need to obtain the final result, but the procedure needs to be sufficiently clear. **(1 point)**.

   We have to apply the equation $\sum \boldsymbol{F} = m\boldsymbol{a_G}$. The forces acting on the body are the weight, -mg$\boldsymbol{k}$ and the reaction $\boldsymbol{N} = N_x\boldsymbol{i} + N_y\boldsymbol{j} + N_z\boldsymbol{k}$. The acceleration of G is obtained by taking the second derivative of the position vector

$$\boldsymbol{r_G} = \ell \boldsymbol{k_2} = \ell(\sin\theta\sin\psi\boldsymbol{i} - \sin\theta\cos\psi\boldsymbol{j} + \cos\theta\boldsymbol{k}). \tag{33}$$

   This leads to 3 trivial equations with 3 unknowns, N_x, N_y and N_z.


$z \equiv z_1$
$z_2 \equiv z_B$
θ
g
G
O
ψ
x
y
$x_1 \equiv x_2$