Please answer the questions 1-6 with True or False and provide a short explanation for your answer. **A right answer without explanation will not be considered.** Solve also exercises 7-8.

1. A pilot wants to perform with her aircraft a steady turn of radius R in a horizontal plane. The aerodynamic efficiency of the aircraft is $E = L/D = 12$, defined as the ratio of the lift to the drag. She doubts between 2 possibilities. The first one is to roll the aircraft with an angle μ and use some of the lift to perform the turn. The second one is to use the rudder to rotate the aircraft a certain yaw angle β and use some of the thrust to perform the turn. After thinking for some time she decides that the second option is worse and selects the first option. Her choice is correct. (True/False)

   The statement is True. In straight level flight the thrust is equal to the drag, and the lift is equal to the weight. Since $E = L/D = 12$ then $L \gg D$ and $L \gg T$. Then, it is generally easier to perform the turn using the lift than using the thrust.

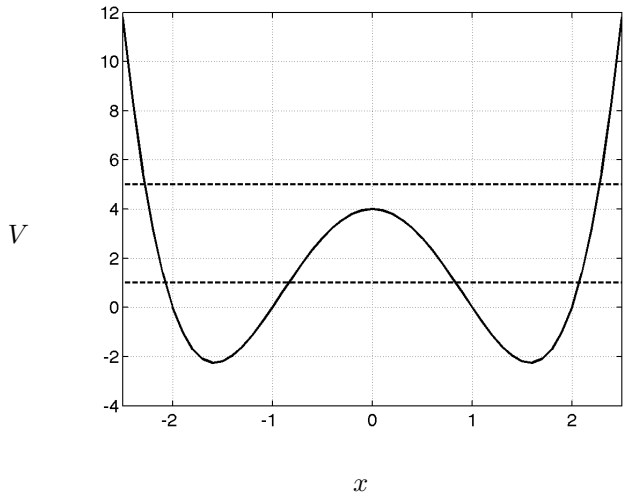
2. A point particle is travelling at constant speed along an elliptical path given by equations

$$x = a\cos\theta,$$
$$y = b\sin\theta,$$

   with $a \neq b$. The tangential acceleration cannot be constant in time. (True/False)

   The statement is False. By definition the tangential acceleration is the time rate of change of the speed. Since the speed is constant the tangential acceleration is zero, so it is constant.

3. A point particle is constrained to move along the $x-$axis under the potential V shown in the figure.



   - At $t = 0$, the particle is released from $x = 0$ towards the right with energy $E = 1$. The subsequent motion is an oscillation between two positions. (True/False)

     The statement is False. It is impossible to release a particle from $x = 0$ with $E = 1$, since $V(0) > 1$. This would imply a negative kinetic energy for the particle, which is unphysical.

   - At $t = 0$, the particle is released from $x = 1.5$ towards the right with energy $E = 1$. The subsequent motion is an oscillation between two positions. (True/False)

     The statement is True. There are four intersections between the curve $E = 1$ and the curve of the potential $V(x)$, however since the particle is released from $x = 1.5$ the particle will oscillate between the two intersections that are located in the positive part of the axis.

   - At $t = 0$, the particle is released from $x = 0$ towards the right with energy $E = 5$. The particle moves towards $x \to \infty$. (True/False)

     The statement is False. The particle will oscillate between the two locations where $V(x) = 5$.

4. Consider a damped harmonic oscillator of mass m_1, spring constant k_1 and damper constant λ_1. Consider also a second oscillator whose corresponding properties are $m_2 = 4m_1$, $k_2 = k_1$ and $\lambda_2 = 4\lambda_1$. If it is true that the solution for the second harmonic oscillator corresponds to the case of large damping, then the solution for the first harmonic oscillator will also correspond to the case of large damping. (True/False)

   The statement is False. The damping parameter for both cases is the same

$$\gamma_1 = \frac{\lambda_1}{2m_1},$$
$$\gamma_2 = \frac{\lambda_2}{2m_2} = \frac{4\lambda_1}{8m_1} = \gamma_1.$$

   The natural frequency is different

$$\omega_1 = \sqrt{\frac{k_1}{m_1}},$$
$$\omega_2 = \sqrt{\frac{k_2}{m_2}} = \sqrt{\frac{k_1}{4m_1}} = \frac{\omega_1}{2}.$$

   We are told that in case 2 the damping is large so that $\gamma_2 > \omega_2$. Since $\omega_1 > \omega_2$ and $\gamma_1 = \gamma_2$, in general we do not know if $\gamma_1 > \omega_1$ or $\gamma_1 < \omega_1$.

5. A point particle P is subject to a central force of magnitude $F(r)$ directed towards a fixed point A. The function $F(r)$ is, however, unknown. In spite of that, we know that the motion of the particle will be contained in a plane and the vector $\boldsymbol{AP}$ will sweep equal areas in equal times. (True/False)

   The statement is True. Since the particle is subject to a central force, we can show that the angular momentum is constant and therefore the trajectory of the particle is contained in a plane. In order to proof that the vector $\boldsymbol{AP}$ sweeps equal areas in equal times we do not need to know the function $F(r)$. See page 54 of the lecture notes.

6. Consider two points P and Q of a rigid body, which are separated by a distance d. P is a fixed point and the body is in a general rotation with an angular velocity $\boldsymbol{\omega}$ with respect to P. By general rotation we mean that the 3 Euler angles are arbitrary functions of time $\psi(t)$, $\theta(t)$, $\varphi(t)$. In that case the velocity of Q cannot be zero. (True/False).

   The statement is False. Using the velocity field of the rigid body we can write

$$\boldsymbol{v_Q} = \boldsymbol{v_p} + \boldsymbol{\omega} \times \boldsymbol{PQ} = \boldsymbol{\omega} \times \boldsymbol{PQ}.$$

   Therefore if the vector $\boldsymbol{PQ}$ is parallel to $\boldsymbol{\omega}$ then the velocity of Q is zero.

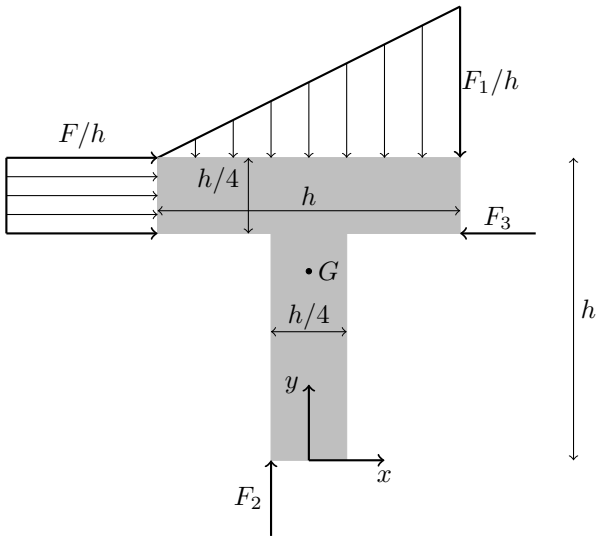
7. Consider a homogeneous planar body in T-shape. The height of the body is h. The width is equal to the height. The thickness in the top part is $h/4$. The thickness in the central part is also equal to $h/4$. The body is subject to a system of forces as indicated in the figure below. Some of the forces are point forces and some of the forces are distributed forces.

   - Determine the position of the centre of mass of the body, G.
     By symmetry considerations it is clear that the centre of mass is located in the symmetry plane, so that $x_G = 0$. The vertical location is given by $y_G = \frac{37}{56}h$, since this is an ugly number, in the following we employ $A = 37/56$, so that $y_G = Ah$.
   - With respect to the point G, determine the resultant force $\boldsymbol{R}$ and the resultant moment $\boldsymbol{M}_G$.
     The 3 components are given by

$$R_x = \boldsymbol{R} \cdot \boldsymbol{i} = \frac{F}{4} - F_3$$
$$R_y = \boldsymbol{R} \cdot \boldsymbol{j} = F_2 - \frac{F_1}{2}$$
$$M_G = \boldsymbol{M_G} \cdot \boldsymbol{k} = \left(\frac{3}{4} - A\right) F_3 h - \frac{1}{8} F_2 h - \frac{1}{12} F_1 h - \frac{1}{4}\left(\frac{7}{8} - A\right) F h$$

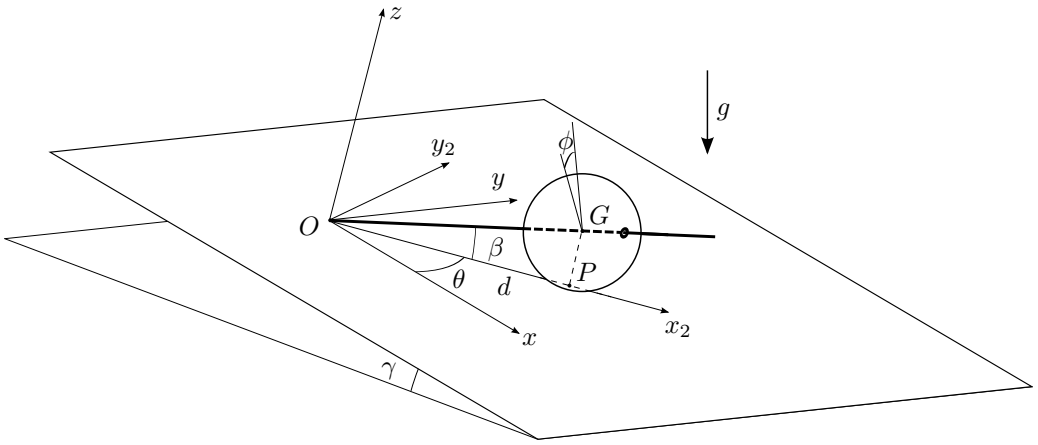
   - Determine the values of F_1, F_2 and F_3 as a function of F and h, so that the system is in equilibrium.
     In equilibrium, $\boldsymbol{R} = 0$ and $\boldsymbol{M}_G = 0$. Solving these equations we obtain

$$F_1 = -\frac{3}{14} F$$
$$F_2 = -\frac{3}{28} F$$
$$F_3 = \frac{1}{4} F$$



8. A heavy, solid, homogeneous sphere of mass M and radius R moves on top of an inclined plane that forms an angle γ with the horizontal. The sphere is always in contact with the inclined plane. The contact point is P and the centre of mass of the sphere is G. A stiff rod of negligible mass traverses the sphere through a small diametral perforation. The sphere can slide along and rotate about the rod freely, without any friction. One end of the rod is fixed at point O of the inclined plane. The rod is free to turn in any direction but it does not rotate along its longitudinal axis.

   Choose the minimum number of variables (linear or angular coordinates) in order to fully determine the position and orientation of the sphere. (Note that the solution is not unique and it is enough if you provide one set of variables).



   We only need 3 variables as indicated in the figure, they could be for example θ, β and ϕ.

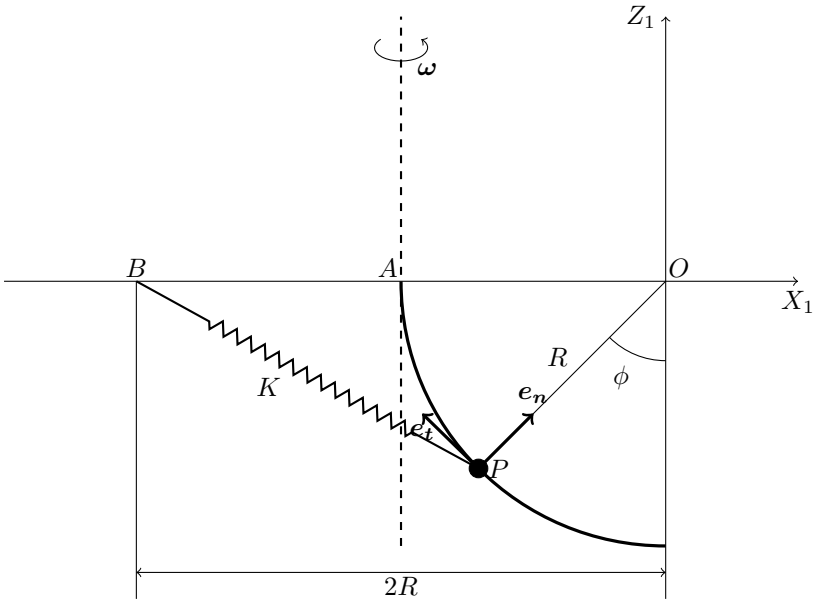
Problem 1

A point particle P of mass m is constrained to move in a smooth quarter of circumference as shown in the figure below. We define a moving reference frame $OX_1Y_1Z_1$ that rotates with respect to an inertial reference frame $AXYZ$. The rotation of the reference frame is performed along a vertical axis defined as $x_1 = -R$ and $y_1 = 0$, the angular velocity is $\boldsymbol{\omega} = \omega \boldsymbol{k} = \omega \boldsymbol{k_1}$. The particle is attached to a linear spring with the other end attached to the point B of the reference frame $OX_1Y_1Z_1$ with coordinates (-2R, 0, 0). The constant of the spring is $K = \alpha m \omega^2$. The angle between $\overline{OP}$ and the negative side of the z_1 axis, ϕ, is used to define the position of the particle. Neglect the action of the gravity on the particle.

Initially the particle is located at $\phi = 0$ and it is set to motion with a relative velocity $-\beta\omega R \boldsymbol{i_1}$. Both α and β are positive nondimensional parameters.

1. Compute the relative velocity and the relative acceleration of the particle in the path variable reference frame $\{\boldsymbol{e_t}, \boldsymbol{e_n}, \boldsymbol{e_b}\}$ as functions of the coordinate ϕ and their derivatives.

The path variable reference frame is shown at the figure below.



Let's start writing the components of path variable reference frame as a function of the $OX_1Y_1Z_1$ variables:

$$\begin{aligned} \boldsymbol{e_t} &= -\cos\phi \boldsymbol{i_1} + \sin\phi \boldsymbol{k_1} \\ \boldsymbol{e_n} &= \sin\phi \boldsymbol{i_1} + \cos\phi \boldsymbol{k_1} \\ \boldsymbol{e_b} &= \boldsymbol{j_1} \end{aligned} \tag{1}$$

It may be useful to obtain also the inverse relations:

$$\begin{aligned} \boldsymbol{i_1} &= \sin\phi \boldsymbol{e_n} - \cos\phi \boldsymbol{e_t} \\ \boldsymbol{k_1} &= \sin\phi \boldsymbol{e_t} + \cos\phi \boldsymbol{e_n} \end{aligned} \tag{2}$$

In this reference frame, the relative position, relative velocity and relative acceleration are computed as:

$$\begin{aligned} \boldsymbol{r'_p} &= -R\boldsymbol{e_n} \\ \boldsymbol{v'_p} &= R\dot{\phi}\boldsymbol{e_t} \\ \boldsymbol{a'_p} &= R\ddot{\phi}\boldsymbol{e_t} + R\dot{\phi}^2\boldsymbol{e_n} \end{aligned} \tag{3}$$

2. Write down the three components of Newton's second law expressed in the path variables reference frame $\{\boldsymbol{e_t}, \boldsymbol{e_n}, \boldsymbol{e_b}\}$.

As the reference frame is rotating, the forces of inertia need to be taken into account, Newton's second law then becomes:

$$\sum \boldsymbol{F_{ext}} + \boldsymbol{F_I} = m\boldsymbol{a'_p} \tag{4}$$

The external forces acting on the particle are:

- *Spring*

$$\boldsymbol{F_s} = -K\boldsymbol{BP} = -K\left(\boldsymbol{BO} + \boldsymbol{OP}\right) = -K\left(2R\boldsymbol{i_1} - R\boldsymbol{e_n}\right) \tag{5}$$

and then:

$$\boldsymbol{F_s} = -K\left(2R\sin\phi - R\right)\boldsymbol{e_n} + 2RK\cos\phi\boldsymbol{e_t} \tag{6}$$

- *Reaction of the circumference:*

$$\boldsymbol{N} = \lambda\boldsymbol{e_n} + \mu\boldsymbol{e_b} \tag{7}$$

The forces of inertia appear due to the rotation of the reference frame. The frame is rotating with an angular velocity $\boldsymbol{\omega} = \omega\boldsymbol{k} = \omega\boldsymbol{k_1}$ and can be computed as:

$$\boldsymbol{F_I} = -m\left(\boldsymbol{a_O} + \boldsymbol{\alpha}\times\boldsymbol{r'_p} + \boldsymbol{\omega}\times\left(\boldsymbol{\omega}\times\boldsymbol{r'_p}\right) + 2\boldsymbol{\omega}\times\boldsymbol{v'_p}\right) \tag{8}$$

where:

$$\begin{aligned} -m\boldsymbol{a_O} &= m\omega^2 R\boldsymbol{i_1} = m\omega^2 R\left(\sin\phi\boldsymbol{e_n} - \cos\phi\boldsymbol{e_t}\right) \\ -m\boldsymbol{\alpha}\times\boldsymbol{r'_p} &= 0 \\ -m\boldsymbol{\omega}\times\left(\boldsymbol{\omega}\times\boldsymbol{r'_p}\right) &= mR\omega\sin\phi\left(\omega\cos\phi\boldsymbol{e_t} - \omega\sin\phi\boldsymbol{e_n}\right) \\ -2m\boldsymbol{\omega}\times\boldsymbol{v'_p} &= 2mR\dot{\phi}\omega\cos\phi\boldsymbol{e_b} \end{aligned} \tag{9}$$

So the Newton law in projected in each one of the variables states:

$$\boldsymbol{e_t} \;:\; 2KR\cos\phi - m\omega^2 R\cos\phi\left(1-\sin\phi\right) = mR\ddot{\phi} \tag{10}$$

$$\boldsymbol{e_n} \;:\; -K\left(2R\sin\phi - R\right) + m\left[\omega^2 R\sin\phi\left(1+\sin\phi\right)\right] + \lambda = mR\dot{\phi}^2 \tag{11}$$

$$-\boldsymbol{e_n} \;:\; \mu + 2mR\dot{\phi}\omega\cos\phi = 0 \tag{12}$$

3. Express the energy conservation equation in the relative reference frame. Discuss if this equation is equivalent to one of the above ones.

The kinetic energy of the particle in the relative reference frame is:

$$T = \frac{1}{2}m\left|\boldsymbol{v'_p}\right|^2 = \frac{1}{2}mR^2\dot{\phi}^2 \tag{13}$$

The potential energy comes from two different sources: the spring and the centrifugal force. Note that Coriolis and the reaction of the circumference do not produce work. The potential energy of the spring is computed as:

$$V_{spring} = \frac{1}{2}K\left|\boldsymbol{BP}\right|^2 = \frac{1}{2}KR^2\left[5 - 4\sin\phi\right] \tag{14}$$

To compute the potential energy of the centrifugal force we need to consider the distance to the axis of rotation $d = R\left(1-\sin\phi\right)$. Then the potential energy is defined as:

$$V_{cent} = -m\frac{1}{2}\omega^2 R^2\left(1-\sin\phi\right)^2 \tag{15}$$

Then the energy conservation equation in the relative frame is:

$$\frac{1}{2}mR^2\dot{\phi}^2 + \frac{1}{2}KR^2\left[5-4\sin\phi\right] - m\frac{1}{2}\omega^2 R^2\left(1-\sin\phi\right)^2 = E_0 = \frac{1}{2}m\beta^2\omega^2 R^2 + \frac{5}{2}KR^2 - m\frac{1}{2}\omega^2 R^2 \tag{16}$$

Lets now consider a different problem. Depending on the values of the parameters, we may find the equilibrium of the forces at different positions.

4. Obtain the values of α and β to have an equilibrium position when the particle is positioned at ϕ equal to $\frac{\pi}{6}$.

Imposing that $\phi = \frac{\pi}{6}$ in eq.(10), considering $K = \alpha m\omega^2$ and imposing acceleration equals to zero:

$$2\alpha m\omega^2 R\frac{\sqrt{3}}{2} - m\omega^2 R\frac{\sqrt{3}}{2}\left(1 - \frac{1}{2}\right) = 0 \tag{17}$$

so to have an equilibrium position at $\phi = \frac{\pi}{6}$, $\alpha = \frac{1}{4}$.

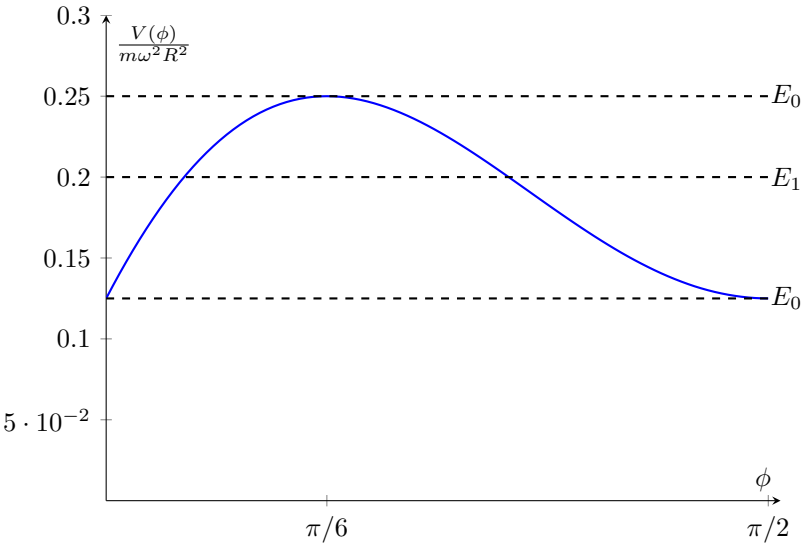
For this value of α, the particle is again located at $\phi = 0$ and is set to motion with the relative velocity defined before.

5. Describe the possible types of motion for the different values of β. In particular, obtain the critical value of β that makes the particle reach the position $\phi = \frac{\pi}{2}$ and the reaction force of the curve to the particle at that instant.

The potential energy in the relative reference frame as a function of ϕ is obtained adding eq.(14) and eq.(15), so:

$$\frac{V(\phi)}{m\omega^2 R^2} = \frac{1}{8}\left[5 - 4\sin\phi\right] - \frac{1}{2}\left(1 - \sin\phi\right)^2 \tag{18}$$

The potential energy is plotted below:



It is possible to identify 3 possible regions:

- *$E < E_0 = \frac{1}{8}m\omega^2R^2$: the motion is not possible.*
- *$E_0 < E < E_2 = \frac{1}{4}m\omega^2R^2$: the particle does not reach the equilibrium position ($\phi = \frac{\pi}{6}$).*
- *$E > E_2 = \frac{1}{4}m\omega^2R^2$: The particle reaches $\phi = \frac{\pi}{2}$ and leaves the circumference arc.*

Therefore the minimum energy required to reach $\phi = \frac{\pi}{2}$ will be E_2. This means that the initial kinetic energy has to be at least $T = \frac{1}{8}m\omega^2R^2 = \frac{1}{2}m\omega^2R^2\beta^2$, so $\beta = \frac{1}{2}$. The reaction forces of the circumference when the particle reaches $\phi = \frac{\pi}{2}$ can be obtained substituting in eq. 11 and 12 the current conditions, obtaining:

$$\begin{aligned} \lambda &= \frac{5}{2}m\omega^2 R \\ \mu &= 0 \end{aligned}$$

6. For a value of β higher than the critical one, the particle leaves the arc of circumference and its motion is not anymore constrained. Under these conditions compute the maximum height reached by the particle, z_{max}, as a function of β. Note that the particle is still under the action of the spring.

When the particle leaves the quarter of circumference is only under the effect of the spring (and the forces of inertia). Taking a look at the vertical direction, ($\mathbf{k_1}$), the only force acting on the particle is the spring, so the equation becomes:

$$m\ddot{z_1} + \frac{1}{4}m\omega^2 z_1 = 0 \tag{19}$$

That is the equation of the harmonic oscillator. The solution of the equation with initial conditions $z_1(0) = 0$ and $\dot{z_1}(0) = \beta\omega R$ is:

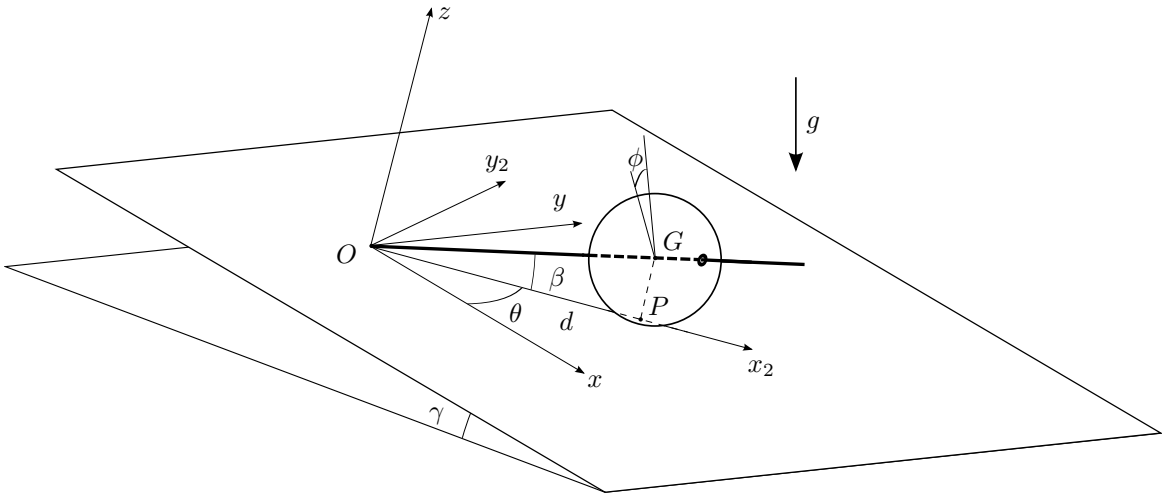
$$z_1(t) = 2\beta R \sin\left(\frac{\omega}{2}t\right) \tag{20}$$

So the maximum height will be $z_{max} = 2\beta R$.

Problem 2

A heavy, solid, homogeneous sphere of mass M and radius R moves on top of an inclined plane that forms an angle γ with the horizontal. The sphere is always in contact with the plane. A stiff rod of negligible mass traverses the sphere through a small diametrical perforation. The sphere can slide along and rotate about the rod freely, without any friction. The interaction between the rod and the sphere can be reduced to a resultant force and a resultant torque with respect to the centre of mass of the sphere, G. These are unknown reactions until the problem is solved. One end of the rod is fixed at point O of the inclined plane. The rod is free to turn in any direction but it does not rotate along its longitudinal axis.

To study the motion of the sphere, we define the inertial reference frame $S\{Oxyz\}$ with origin at O, the x axis pointing downward along the maximum slope line of the plane, and the z axis pointing upward perpendicular to the plane. We also define an auxiliary rotating reference frame $S_2\{Ox_2y_2z_2\}$, where x_2 always contains the point P of contact of the sphere with the plane, and $z_2 \equiv z$. In order to describe the state of the system, we will use the angle θ between x and x_2, the distance $d = |\overline{OP}|$, and the rotation angle ϕ of the sphere about the rod. For convenience, we also define the angle β between the plane and the rod. Note that $\tan\beta = R/d$.



You are asked to:

1. Write down the general expression for the angular velocity and angular acceleration of the sphere ($\boldsymbol{\omega}$ and $\boldsymbol{\alpha}$), and the velocity and acceleration of G ($\boldsymbol{v}_G$ and $\boldsymbol{a}_G$), as a function of θ, d, ϕ and their derivatives.

We will first consider that the friction of the sphere with the inclined plane is infinite, so that it cannot slip with respect to the plane.

2. Relate the velocity of the center of mass G and the velocity of P, and use this to express the time derivatives $\dot{d}$ and $\dot{\phi}$ in terms of $\dot{\theta}$ and the angle β formed by the rod with the plane.

3. Identify all the components of reaction forces and reaction torques acting on the rod alone. Using the balance of forces and torques on the rod, express all reaction forces and torques as a function of only two force components. Hint: it will be easier if you use a new reference frame centered in O with one axis pointing in the direction of the rod. Also, since the rod is massless, it must always be in static equilibrium, so the sum of forces and the sum of torques in each direction must be zero.

4. Identify all the components of forces and torques acting on the sphere, and reduce them to a sum of forces and a sum of torques on G in terms of the unit vectors of S_2. Hint: using the relations derived in the previous point, there are only 5 independent unknowns.

5. Obtain a closed set of six differential equations that would provide the evolution of θ and the 5 unknowns mentioned in the previous question. Note that you do not need to solve any of these equations.

6. If the system is in equilibrium when $\theta = 0$, then compute the reaction force between the rod and the sphere. Compute also the normal reaction and the friction force between the sphere and the inclined plane.

Consider now that the inclined plane is frictionless instead:

7. What changes in this case? Adapt the differential equations to the new case.

8. If the center of the sphere is initially located at $\boldsymbol{r}_{G0} = 2R\boldsymbol{i} + 3R\boldsymbol{j} + R\boldsymbol{k}$ and has an initial velocity $\boldsymbol{v}_{G0} = -\sqrt{Rg}(\boldsymbol{i}+\boldsymbol{j})$, derive the initial conditions that are necessary to integrate the differential equations.

Note: the moment of inertia of a homogeneous sphere of mass M and radius R with respect to an axis that passes through its center of mass is $I = 2MR^2/5$. Remember to define any unit vector that you use in terms of the unit vectors of S ($\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$).

Preliminaries:

We will first express the unit vectors of S_2 in terms of the unit vectors of S as

$$\boldsymbol{i}_2 = \cos\theta\boldsymbol{i} + \sin\theta\boldsymbol{j}; \quad \boldsymbol{j}_2 = -\sin\theta\boldsymbol{i} + \cos\theta\boldsymbol{j}; \quad \boldsymbol{k}_2 = \boldsymbol{k}. \tag{21}$$

Also, since it will be useful for later, we will define the reference frame $S_3\{Ox_3y_3z_3\}$ that results from rotating S_2 an angle β about $-\boldsymbol{j}_2$ (i.e., the x_3 axis aligned with the rod):

$$\boldsymbol{i}_3 = \cos\beta\boldsymbol{i}_2 + \sin\beta\boldsymbol{k}_2; \quad \boldsymbol{j}_3 = \boldsymbol{j}_2; \quad \boldsymbol{k}_3 = -\sin\beta\boldsymbol{i}_2 + \cos\beta\boldsymbol{k}_2. \tag{22}$$

Now that we have defined all these unit vectors, we may use them to write down our answers.

Kinematics:

Angular velocity and acceleration:

$$\boldsymbol{\omega} \equiv \boldsymbol{\omega}_{B/1} = \boldsymbol{\omega}_{B/2} + \boldsymbol{\omega}_{2/1} = \dot{\phi}(\cos\beta\boldsymbol{i}_2 + \sin\beta\boldsymbol{k}_2) - \dot{\beta}\boldsymbol{j}_2 + \dot{\theta}\boldsymbol{k}_2, \tag{23}$$

$$\boldsymbol{\alpha} \equiv \left.\frac{d\boldsymbol{\omega}}{dt}\right|_1 = \ddot{\phi}(\cos\beta\boldsymbol{i}_2 + \sin\beta\boldsymbol{k}_2) + \dot{\phi}\dot{\beta}(-\sin\beta\boldsymbol{i}_2 + \cos\beta\boldsymbol{k}_2) + \dot{\phi}\dot{\theta}\cos\beta\boldsymbol{j}_2 - \ddot{\beta}\boldsymbol{j}_2 + \dot{\beta}\dot{\theta}\boldsymbol{i}_2 + \ddot{\theta}\boldsymbol{k}_2. \tag{24}$$

Center of mass velocity and acceleration:

$$\boldsymbol{v}_G = \dot{d}\boldsymbol{i}_2 + \dot{\theta}d\boldsymbol{j}_2, \tag{25}$$

$$\boldsymbol{a}_G = (\ddot{d} - \dot{\theta}^2 d)\boldsymbol{i}_2 + (2\dot{d}\dot{\theta} + \ddot{\theta}d)\boldsymbol{j}_2. \tag{26}$$

No slip case

We begin analyzing the case with no slip with respect to the inclined plane. This means that the point P of the sphere that is in contact with the plane at a given instant of time must have zero velocity, $\boldsymbol{v}_P = 0$. Using this fact, and relating the velocity at P and G with the equation of the field of velocities of a rigid body,

$$\boldsymbol{v}_G + \boldsymbol{\omega} \times \overline{GC} = (\dot{d} + R\dot{\beta})\boldsymbol{i}_2 + (\dot{\theta}d + R\dot{\phi}\cos\beta)\boldsymbol{j}_2 = \boldsymbol{0} \tag{27}$$

From the $\boldsymbol{i}_2$ component,

$$\dot{d} = 0; \quad \dot{\beta} = 0. \tag{28}$$

From the $\boldsymbol{j}_2$ component,

$$\dot{\phi} = -\frac{\dot{\theta}}{\sin\beta} \tag{29}$$

(we have used the relation between d and β mentioned in the problem). Clearly, these expressions reveal that β (and d) is constant in this case, and that we only need one variable, θ, to describe the state of the system (i.e., there is only one actual degree of freedom in this case).

Analysis of the reactions on the rod:

The rod is massless. That means that all forces and torques must cancel out, always (otherwise, an *infinite* acceleration—linear or angular—would appear).

On the constraint on point O, the inclined plane creates a reaction force on the rod:

$$\boldsymbol{F}_O = F_y \boldsymbol{j}_3 + F_z \boldsymbol{k}_3. \tag{30}$$

Note that the force component along $\boldsymbol{i}_3$ must be zero, since it is the only force on the rod in that direction (it is fine to write it down now, but we would find that it is zero nevertheless, when we look into the force balance). Likewise, due to similar arguments, there is no torque along $\boldsymbol{i}_3$ on the rod at O.

The sphere causes a distribution of forces over the rod where they are in contact. As said in the problem, we can always reduce a system of forces to a resultant force and torque on a point, e.g. G. Let's write this force and torque as

$$\boldsymbol{F}_S = S_y \boldsymbol{j}_3 + S_z \boldsymbol{k}_3, \tag{31}$$
$$\boldsymbol{M}_S = Q_y \boldsymbol{j}_3 + Q_z \boldsymbol{k}_3. \tag{32}$$

The equilibrium of forces on the rod dictates that

$$F_y + S_y = 0, \tag{33}$$
$$F_z + S_z = 0, \tag{34}$$

which reveals that only two force components are independent. Lastly, we also write down the equilibrium of moments of forces on the rod e.g. at O:

$$Q_z + \frac{R}{\sin\beta} S_y = 0, \tag{35}$$
$$Q_y - \frac{R}{\sin\beta} S_z = 0, \tag{36}$$

meaning that we can also express Q_y and Q_z in terms of the two independent forces.

Analysis of the forces and torques on the sphere:

Weight of the sphere: it can be reduced to a single force on G, no torque:

$$\boldsymbol{W} = -Mg\cos\gamma \boldsymbol{k} + Mg\sin\gamma \boldsymbol{i} = -Mg\cos\gamma \boldsymbol{k}_2 + Mg\sin\gamma(\cos\theta \boldsymbol{i}_2 - \sin\theta \boldsymbol{j}_2) \tag{37}$$

Inclined plane reaction on the sphere: The point of contact, P, receives a reaction force. We may write it down as follows. When we reduce the force to a force and torque on G, we obtain:

$$\boldsymbol{F}_N = N_x \boldsymbol{i}_2 + N_y \boldsymbol{j}_2 + N_z \boldsymbol{k}_2 \tag{38}$$
$$\boldsymbol{M}_N = RN_y \boldsymbol{i}_2 - RN_x \boldsymbol{j}_2 \tag{39}$$

Rod reaction on the sphere: it is simply the same force and torque on G that the sphere was causing on the rod, but with opposite sign:

$$\boldsymbol{F}_R = -S_y \boldsymbol{j}_3 - S_z \boldsymbol{k}_3 = -S_y \boldsymbol{j}_2 + S_z(\sin\beta \boldsymbol{i}_2 - \cos\beta \boldsymbol{k}_2), \tag{40}$$
$$\boldsymbol{M}_R = -Q_y \boldsymbol{j}_3 - Q_z \boldsymbol{k}_3 = -\frac{RS_z}{\sin\beta} \boldsymbol{j}_2 + \frac{RS_y}{\sin\beta}(-\sin\beta \boldsymbol{i}_2 + \cos\beta \boldsymbol{k}_2) \tag{41}$$

To obtain the 6 equations of motion that we seek, we will express the linear and angular momentum equations applied to the sphere, projecting them along $\boldsymbol{i}_2$, $\boldsymbol{j}_2$, and $\boldsymbol{k}_2$. Note: exactly the same result is reached if these equations are applied to the whole system (rod + sphere), taking moments about O. when only external forces/torques are considered.

Forces on G:

$$-\frac{MR}{\tan\beta}\dot{\theta}^2 = Mg\sin\gamma\cos\theta + N_x + S_z\sin\beta \tag{42}$$
$$\frac{MR}{\tan\beta}\ddot{\theta} = -Mg\sin\gamma\sin\theta + N_y - S_y \tag{43}$$
$$0 = -Mg\cos\gamma + N_z - S_z\cos\beta \tag{44}$$

Moment of forces on G:

$$\frac{2MR}{5\tan\beta}\ddot{\theta} = N_y - S_y, \tag{45}$$

$$\frac{2MR}{5\tan\beta}\dot{\theta}^2 = -N_x - \frac{S_z}{\sin\beta}, \tag{46}$$

$$\frac{4MR\tan\beta}{5}\ddot{\theta} = S_y. \tag{47}$$

Observe that, while it was not part of the problem, we can get an equation for the evolution of θ as a linear combination of these equations that is independent of the reactions. Also, in this no-slip case, the energy equation would also give us the evolution of θ.

Equilibrium position

In equilibrium, all time derivatives are zero. Solving the system for $\dot{\theta} = 0$ and $\ddot{\theta} = \ddot{\phi} = 0$:

$$N_y = S_y = 0, \tag{48}$$

$$S_z = \frac{Mg\sin\gamma\sin\beta}{\cos^2\beta}, \tag{49}$$

$$N_x = -\frac{Mg\sin\gamma}{\cos^2\beta}, \tag{50}$$

$$N_z = Mg\cos\gamma + Mg\sin\gamma\tan\beta. \tag{51}$$

It is interesting to analyze the behavior of these forces for $\beta \ll 1$ and $(\pi/2 - \beta) \ll 1$ (but this was not part of the problem).

Frictionless case:

The only thing that changes in this case is that there is no friction force ($N_x = N_y = 0$), and that the full three degrees of freedom of the problem have to be considered (the motion is no longer a function of θ only).

The new equations are:

Forces on G:

$$M(\ddot{d} - \dot{\theta}^2 d) = Mg\sin\gamma\cos\theta + S_z\sin\beta \tag{52}$$

$$M(2\dot{d}\dot{\theta} + \ddot{\theta}d) = -Mg\sin\gamma\sin\theta - S_y \tag{53}$$

$$0 = -Mg\cos\gamma + N_z - S_z\cos\beta \tag{54}$$

Moment of forces on G:

$$\frac{2MR}{5}(\ddot{\phi}\cos\beta - \dot{\phi}\dot{\beta}\sin\beta + \dot{\beta}\dot{\theta}) = -S_y, \tag{55}$$

$$\frac{2MR}{5}(\dot{\phi}\dot{\theta}\cos\beta - \ddot{\beta}) = -\frac{S_z}{\sin\beta}, \tag{56}$$

$$\frac{2MR}{5}(\ddot{\phi}\sin\beta + \dot{\phi}\dot{\beta}\cos\beta + \ddot{\theta}) = \frac{S_y}{\tan\beta}. \tag{57}$$

For the initial conditions, we identify in the data that $\theta_0 = \arctan(3/2)$, and $d_0 = \sqrt{13}R$. Projecting the initial velocity in axes S_2, we can see that $\dot{d}_0 = -\sqrt{Rg}(\cos\theta_0 + \sin\theta_0)$, and $\dot{\theta}_0 = -\sqrt{Rg}(\cos\theta_0 - \sin\theta_0)/d_0$. No indication is given about the rotation of the sphere about the rod, so we will just take $\phi_0 = 0$ initially and leave $\dot{\phi}_0$ as an unknown parameter.