THEORY

Please answer the questions 1-7. Those questions requiring a TRUE/FALSE answer also require a brief explanation. **A right answer without explanation will not be graded**.

1. A heavy UAV is flying when it runs out of fuel. At that point it has a mass m. The UAV then descends at a constant angle (which may not be small) and with constant velocity. Recalling that the magnitude of the lift and drag forces is $L = \frac{1}{2}\rho v^2 c_L S$ and $D = \frac{1}{2}\rho v^2 c_D S$, write down an expresion for the flight path angle γ as a function of the lift and drag coefficients, c_L and c_D, and an expression for the velocity v as a function of γ and the parameters of the problem.

   Solution:

$$\tan\gamma = c_D/c_L \tag{1}$$

$$v^2 = 2mg/[\rho S(c_L\cos\gamma + c_D\sin\gamma)] \tag{2}$$

2. If a particle P is moving with constant radius of curvature ρ, then its trajectory is a circumference (TRUE/FALSE).

   FALSE: that is only true if additionally the normal vector $\boldsymbol{e}_n$ always points to the same point (the center of the circumference). A counterexample is a helix of constant pitch (has constant ρ but the motion is clearly not planar).

3. We have a heavy point mass m hanging from the hook of a construction crane, connected by a spring of stiffness k and a viscous damper with damping constant λ.

   (a) The presence of the gravity force acting in the direction of the spring can change the resonant frequency of the system (TRUE/FALSE).

   FALSE: It affects the position of the equilibrium point (which now requires an elongation of the spring $\ell = mg/k$), but it does not affect the frequency response.

   (b) The crane operator starts moving the hook up and down sinusoidally as $y_h = A\sin\omega t$, where y_h is the vertical position of the hook with respect to a reference height. For the motion of P about its equilibrium position, this is completely equivalent to leaving the hook stationary and applying a vertical force directly at P, equal to $\boldsymbol{F} = (Ak\sin\omega t + A\lambda\omega\cos\omega t)\boldsymbol{j}$ (TRUE/FALSE).

   TRUE: calling y_m the height of the mass P, its motion is given by Newton's second law,

$$m\ddot{y}_m + \lambda(\dot{y}_m - \dot{y}_h) + k(y_m - y_h) = 0, \tag{3}$$

   and we may write

$$m\ddot{y}_m + \lambda\dot{y}_m + ky_m = \lambda\dot{y}_h + ky_h, \tag{4}$$

   so the moving hook appears as a driving force.

   (c) If the frequency of the hook motion is $\omega = \sqrt{\omega_0^2 - \gamma^2}$, where $\omega_0^2 = k/m$ and $\gamma = \lambda/(2m)$, then if we wait long enough, the spring will eventually break because the amplitude of the oscillations of P grows without bound (TRUE/FALSE).

   FALSE: when there is at least some damping, the oscillations do not grow unbounded (although at resonance they can be large).

4. A satellite has set of body axes $GX_1Y_1Z_1$ fixed to it. The satellite performs a 90 deg rotation about its X_1 axis, then a 90 deg rotation about its Y_1 axis, then a 90 deg rotation about its Z_1 axis.

   (a) There does not exist any other set of three rotations that results in the same final orientation (TRUE/FALSE).

   FALSE. We can arrive at that orientation in multiple ways. For example, two rotations of 90 deg about its Z axis and then -90 deg about its Y axis. A sufficient condition for two sequences of rotations to be equivalent in the result is that they reduce to the same single rotation (through Euler's theorem).

   (b) Given two rotations about different axes, we can state that they never commute (TRUE/FALSE).

   FALSE. Take, for instance a 180 deg rotation about axis X and a 180 deg rotation about axis Y. In general, rotations do not commute, though.

5. An amorphous, rigid asteroid is rotating and translating in free space. Using an advanced telescope, we take instantaneous measurements of the velocity of two different points A and B on its surface and express them in a given reference frame $OXYZ$.

   (a) From these two measurements we can determine the angular velocity of the asteroid in $OXYZ$ (TRUE/FALSE).

   FALSE: The component of $\boldsymbol{\omega}$ along AB is undetermined.

   (b) If $\boldsymbol{v}_A \parallel \boldsymbol{v}_B$ and $\boldsymbol{v}_A \neq \boldsymbol{v}_B$, then the motion is a pure rotation (i.e., no slip) (TRUE/FALSE).

   TRUE: since $\boldsymbol{v}_B = \boldsymbol{v}_A + \boldsymbol{\omega} \times AB$, clearly $\boldsymbol{\omega} \times AB \parallel \boldsymbol{v}_A$. $\boldsymbol{\omega}$ is necessarily contained in the plane generated by vectors AB and $\boldsymbol{v}_A \times AB$. Hence, $\boldsymbol{v}_A \cdot \boldsymbol{\omega} = 0$ and there is no slip velocity.

   (c) If we measure $\boldsymbol{v}_A = (1, 0, 1)$ and $\boldsymbol{v}_B = (1, 1, -1)$, with $AB = (1, 2, 0)$, we can state without doubt that the measurement system is defective (TRUE/FALSE).

   TRUE: Since the asteroid is a rigid body, we know that $\boldsymbol{v}_B = \boldsymbol{v}_A + \boldsymbol{\omega} \times AB$. Dot-multiplying by AB we get $\boldsymbol{v}_B \cdot AB = \boldsymbol{v}_A \cdot AB$. This condition is not met by the measured velocities

6. Consider a generic rigid body of mass m moving in space (rotating and translating). To study its motion, we define a set of inertial axes, $OXYZ$, and a set of axes at the center of mass of the body, $GX_GY_GZ_G$, parallel to $OXYZ$.

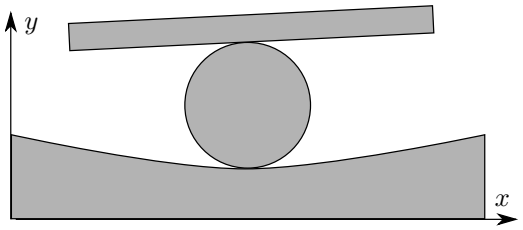
   (a) The kinetic energy of the body with respect to $OXYZ$, is always greater than or equal to the kinetic energy with respect to $GX_GY_GZ_G$, i.e., $T_0 \geq T_G$ (TRUE/FALSE)

   TRUE. The kinetic energy with respect to $OXYZ$ can be written as $T_0 = T_G + \frac{1}{2} m v_G^2$

   (b) The angular momentum of the body with respect to $OXYZ$, is always greater than or equal to the angular momentum with respect to $GX_GY_GZ_G$, i.e., $|\boldsymbol{H}_0| \geq |\boldsymbol{H}_G|$ (TRUE/FALSE)

   FALSE. In this case we have $\boldsymbol{H}_0 = \boldsymbol{H}_G + m\boldsymbol{r}_G \times \boldsymbol{v}_G$, and the two terms add vectorially, so we can have $|\boldsymbol{H}_0| < |\boldsymbol{H}_G|$, $|\boldsymbol{H}_0| = |\boldsymbol{H}_G|$ or $|\boldsymbol{H}_0| > |\boldsymbol{H}_G|$.

7. A rectangular piece of wood is sitting on top of a cylinder as shown in the figure, over which it can roll without slip. Additionally, the cylinder rolls without slip over a curved floor. How many degrees of freedom are needed to fully describe the system? Propose a valid set of generalized coordinates to do so.



   Solution:

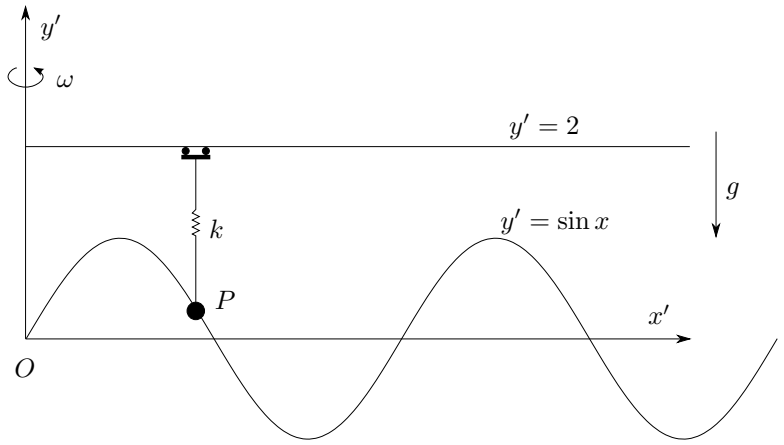
   Since there is no slip, two degrees of freedom (one for the rotation of the block over the cylinder, the other for the rotation of the cylinder over the floor. A valid set of coordinates would be, for instance, the axial x of the center of the cylinder and the angle with respect to the vertical of the contact point of the block with the cylinder.

PROBLEM 1

A heavy bead P of mass m slides without friction along a thin wire with shape given by the following equation:

$$y'(x) = \sin x'. \tag{5}$$

Moreover, the bead is joined to a straight rail located at $y' = 2$ by a spring of stiffness k. The connexion of the spring with the rail can move freely along the rail, so that the spring is always vertical. The whole setup rotates with respect to the y' axis with a constant angular velocity ω. A sketch of the setup and the definition of the right-handed, rotating $Ox'y'z'$ reference frame is given below.



1. Write down the general expression for the components of the relative velocity and acceleration of P in the reference frame $Ox'y'z'$ as functions of the axial position of the bead, x', and its time derivatives $\dot{x}'$ and $\ddot{x}'$.

$$v'_x = \dot{x}' \tag{6}$$
$$v'_y = \dot{y}' = \dot{x}' \cos x' \tag{7}$$
$$v'_z = 0 \tag{8}$$
$$a'_x = \ddot{x}' \tag{9}$$
$$a'_y = \ddot{y}' = -\dot{x}'^2 \sin x' + \ddot{x}' \cos x' \tag{10}$$
$$a'_z = 0 \tag{11}$$

2. Identify all forces acting on the bead and then write down the three components of Newton's second law in the $Ox'y'z'$ reference frame. From these equations, obtain an explicit expression of the components of the wire reaction force as a function of x' $\dot{x}'$ and the parameters of the problem only, and a second order differential equation for x' that does not depend on the reaction force.

   Applied forces are:

$$\boldsymbol{F}_k = (2 - \sin x')k\boldsymbol{j} \tag{12}$$
$$\boldsymbol{F}_g = -mg\boldsymbol{j} \tag{13}$$

   Then, there are the reaction forces of the thin wire (two independent components):

$$\boldsymbol{R} = -N \cos x'\boldsymbol{i} + N\boldsymbol{j} + Q\boldsymbol{k} \tag{14}$$

   And inertia forces in the rotating $Ox'y'z'$ frame are (centrifugal and Coriolis)

$$\boldsymbol{F}_I = m\omega^2 x'\boldsymbol{i} + 2m\omega\dot{x}'\boldsymbol{k} \tag{15}$$

   The Equation of Newton along each direction of $Oxyz$ reads:

$$m\ddot{x}' = m\omega^2 x' - N \cos x' \tag{16}$$
$$m\left(-\dot{x}'^2 \sin x' + \ddot{x}' \cos x'\right) = (2 - \sin x')k - mg + N \tag{17}$$
$$0 = Q + 2m\omega\dot{x}' \tag{18}$$

The reaction forces are therefore given by:

$$N(1+\cos^2 x') = m\left(-\dot{x}'^2 \sin x' + \omega^2 x' \cos x'\right) - (2-\sin x')k + mg \tag{19}$$

$$Q = -2m\omega\dot{x}' \tag{20}$$

Using the first two equations to eliminate N, we obtain a second order differential equation for x:

$$m\ddot{x}'(1+\cos^2 x') - 2m\dot{x}'^2 \sin 2x' - m\omega^2 x' = [(2-\sin x')k - mg]\cos x' \tag{21}$$

3. Is it possible to use the equation of conservation of mechanical energy of the relative motion of P? If so, write it down in terms of x', $\dot{x}'$ and the parameters of the problem. What is the relation of this equation with the differential equation for x' in the previous point?

   Yes, it is possible to use the conservation of energy because all forces that exert work derive from a potential (incl. the centrifugal force).

   Kinetic energy:

$$T = \frac{1}{2}m\dot{x}'^2(1+\cos^2 x') \tag{22}$$

   Potential energy:

$$V = mg\sin x' - \frac{1}{2}k(2-\sin x')^2 - \frac{1}{2}m\omega^2 x'^2 \tag{23}$$

   Hence, the equation of conservation of energy ($T+V=E$) reads:

$$m\dot{x}'^2(1+\cos^2 x') + 2mg\sin x' - k(2-\sin x')^2 - m\omega^2 x'^2 = 2E \tag{24}$$

   As it could be expected, this equation is the integral of Eq. 21. You can verify this by derivation, and then simplifying the factor $2\dot{x}$. The energy equation can therefore be used as an alternative to Eq. 21, and this is a good choice because it is already integrated once. This is generally true in conservative problems with just one degree of freedom, as it is the case here.

4. Discuss the existence of equilibrium positions in the range $x' \in [0, 2\pi[$ when $\omega = 0$ as a function of the parameter $\alpha = k/mg$.

   Equilibrium positions are found in Newton equations by taking $\dot{x}' = 0$ and $\ddot{x}' = 0$. From the first equation, it is clear that $\cos x' = 0$ or $N = 0$ for equilibrium.

   For $\cos x' = 0$ we find two positions regardless of the value of α (the peak and the valley in the sine function):

$$(a) \quad x' = \pi/2; N = mg - k \tag{25}$$

$$(b) \quad x' = 3\pi/2; N = mg - 3k \tag{26}$$

$$\tag{27}$$

   For $N = 0$ we have to solve the following equation:

$$2 - \frac{1}{\alpha} = \sin x'. \tag{28}$$

   - When $\alpha < 1/3$, there are no solutions of this type. *Moreover, a simple analysis would show that equilibrium position* (a) *is unstable while* (b) *is stable.*
   - When $1/3 < \alpha < 1$, one additional equilibrium positions appear in each segment between a peak and a valley or a valley and a peak (let's call this solution (c)). *The solution* (c) *is stable, and* (a) *and* (b) *become unstable.*
   - When $\alpha > 1$, there are no solutions of this type again. *Solution* (a) *is stable, and* (b) *unstable.*

5. Consider now that $\omega \neq 0$, and $\alpha = 0$. We drop the bead P at $x' = \pi$ with no initial velocity. Sketch the shape of the potential and discuss qualitatively the motion of P if the parameter $\beta = \omega^2/g$ is small or large. Write down the system of equations that would allow to solve for the critical value of β that divides between the two behaviors.

   Particularizing the equation of energy for the initial condition and dividing over mg we find that

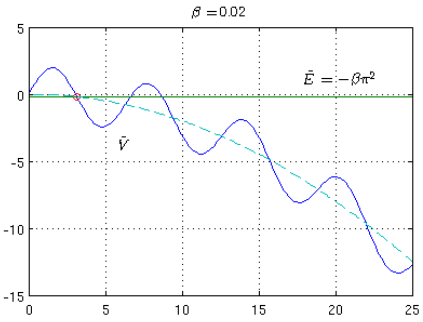
$$\tilde{E} \equiv 2E/(mg) = -\beta\pi^2 \tag{29}$$

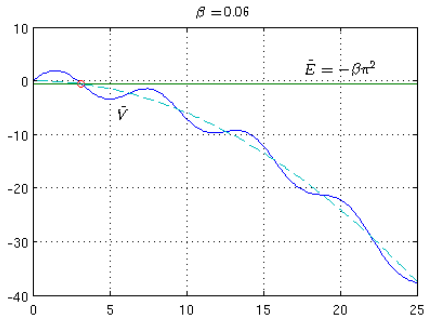
Sketching the shape of the potential $\tilde{V}$ helps to understand the possible motions. $\tilde{V}$ is a parabola with a superposed sinusoidal oscillation. Initially, the particle will roll down to the valley and start climbing up the next peak. The particle will stop and turn around if there is a point where the kinetic energy would cross the zero. We can find these points with the equation of energy by doing $\dot{x}' = 0$:

$$\tilde{V} \equiv 2\sin x' - \beta x'^2 = -\beta\pi^2 \equiv \tilde{E}, \tag{30}$$

(a) If β is small, the next maximum of $\tilde{V}$ is larger than the mechanical energy. The particle will turn back and enter an oscillatory motion between the initial position and that turning point.

(b) If β is large, the energy $\tilde{E}$ is enough to overcome the next maximum in the potential, or the next maximum disappears altogether. Hence, the bead will slip to infinity afterwards.





The critical value of β between the two behaviors can be found imposing that there is a point where velocity goes to zero, so $\dot{x}' = 0$,

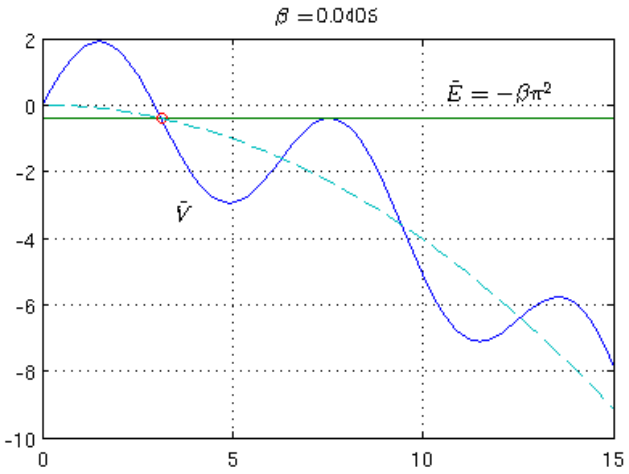
$$2\sin x' - \beta x'^2 = -\beta\pi^2, \tag{31}$$

and that this point coincides with the maximum in the potential,

$$\frac{d\tilde{V}}{dx'} = 2\cos x' - 2\beta x' = 0 \tag{32}$$

Just for completeness: solving numerically these two equations together yields the critical value of β and the position at which the zero velocity point is:

$$\beta^* = 0.0405; \quad x'^* = 7.54 \tag{33}$$

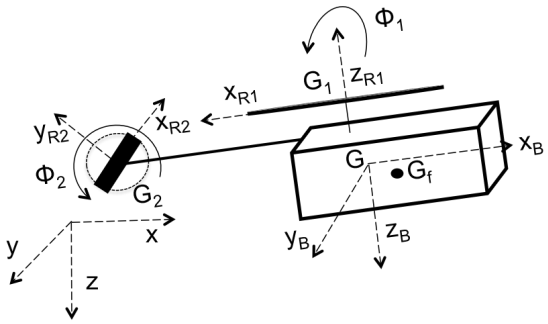


PROBLEM 2

Consider a toy RC rotorcraft as the one shown in the picture. This rotorcraft can be represented as a system of three rigid bodies, namely: the fuselage, the main rotor and the tail rotor.

- The fuselage can be understood as the cabin and a tail boom of negligible mass and length L. For the sake of simplicity, the cabin can be considered as an homogeneous rectangular prism of length $d = L$, width $w = L/10$, height $h = L/10$, and mass $m_f = M$. Consider G_f to be the center of mass of the fuselage.
- The main rotor is modeled as a bar of length $L_{R_1} = 2R$ and mass $m_{R_1} = M/3$. The center of mass of the main rotor (G_1) is located at a distance $3/4L$ from the nose of the cabin and at at a distance $L/10$ above the cabin.
- The tail rotor as a bar of $L_{R_2} = R/5$ and $m_{R_2} = M/15$. The center of mass of the tail rotor (G_2) is placed within the plane $Gx_By_Bz_B$ at the far end of the tail boom and at a distance $L/10$ below the top of the cabin.





To analyze the dynamics of the rotorcraft first we consider an inertial reference frame $Oxyz$ defined in such a way that the plane Oxy is the horizontal plane. Furthermore, a reference frame attached to the fuselage (body-fixed) is considered for the analysis. This reference frame is defined with origin at the center of mass of the system, G, with the axis Gx_B pointing towards the nose of the rotorcraft. Gz_B is defined as the direction perpendicular to Gx_B and directed as shown in the figure. Finally, Gy_B is perpendicular to Gx_B and Gz_B. We define two additional reference frames each moving with one of the rotors $G_ix_{R_i}y_{R_i}z_{R_i}$ ($i = 1$ for the main rotor and $i = 2$ for the one at the tail). Note that the axis $G_ix_{R_i}$ goes in the direction of the respective rotor length.

In the analysis we can consider that the orientation of the fuselage is determined by two angles: α, the angle between Gz_B and Gz, and φ is the yaw angle around Gz_B[1]. Note that roll is not considered in this exercise. We may assume that the main rotor is constrained to rotate in a plane parallel to the Gx_By_B, so its orientation is determined by one angle ϕ_1. On the other hand, the tail rotor is contained in a plane parallel to Gx_Bz_B, but again its orientation is determined by one angle ϕ_2.

A basic aerodynamic model is proposed for both rotors. The main rotor generates a force perpendicular to the rotational plane and its value is:

$$\boldsymbol{F}_{R_1} = -k\left(\dot{\phi_1}R\right)^2 \boldsymbol{k}_B$$

It also creates a torque:

$$\boldsymbol{M}_{R_1} = \frac{k}{50}\dot{\phi_1}^2 R^3 \boldsymbol{k}_B$$

In the case of the tail rotor, the force follows a equivalent law:

$$\boldsymbol{F}_{R_2} = k\left(\dot{\phi_2}\frac{R}{5}\right)^2 \mathbf{j_B}$$

The torque of the second rotor is considered negligible.

[1] It is recommended to use an auxiliary reference frame $x_1y_1z_1$ defined from $Oxyz$ by rotating α around y axis. Then an angle φ around z_1.

Due to the control system of this particular type of rotorcraft it can be shown that the angle α can be considered constant and the force required to move forward is obtained from the projection of the force generated by the main rotor.

In addition, the rotors are considered one-dimensional, and thus their tensor of inertia with respect to each center of mass in reference frame fixed to each rotor $G_i x_{R_i} y_{R_i} z_{R_i}$ equal to:

$$\bar{\bar{I}}_{G_i}^{R_i} = \frac{1}{12} m_{R_i} L_{R_i}^2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Concerning the fuselage the tensor of inertia in the fuselage-fixed reference frame $G_f x_B y_B z_B$ is:

$$\bar{\bar{I}}_{G_f}^{fus} = \frac{1}{12} M L^2 \begin{bmatrix} \frac{w^2}{L^2} + \frac{h^2}{L^2} & 0 & 0 \\ 0 & \frac{d^2}{L^2} + \frac{h^2}{L^2} & 0 \\ 0 & 0 & \frac{d^2}{L^2} + \frac{w^2}{L^2} \end{bmatrix}$$

Gravity must be considered, but we assume that all the other additional external forces and torques not mentioned are negligible.

- Compute the total mass and the position of the center of mass of the system, in particular compute the vector $\mathbf{G_f G}$ in the fuselage fixed reference frame.

  The total mass is $m_t = \frac{7}{5} M$. Considering a reference frame with origin in point G_f, the center of gravity of the fuselage, and parallel to the $G x_B y_B z_B$. The center of gravity of the system will be at a distance $-\frac{11}{84} L$ in x_B direction, and $-\frac{14}{420} L$ in z_B. Thus $\boldsymbol{r}_{G_f G} = -\frac{11}{84} L \boldsymbol{i}_B - \frac{11}{84} \boldsymbol{k}_B$.

- Compute the angular velocity of the fuselage $\boldsymbol{\omega}_f$, the main rotor $\boldsymbol{\omega}_1$, and the tail rotor $\boldsymbol{\omega}_2$ with respect to the inertial reference frame as a function of α, φ, ϕ_1, ϕ_2, $\dot{\varphi}$, $\dot{\phi}_1$, $\dot{\phi}_2$.

  The relationships between xyz and $x_1 y_1 z_1$ is:

$$\begin{aligned} \boldsymbol{i}_1 &= \cos\alpha \boldsymbol{i} - \sin\alpha \boldsymbol{k} \\ \boldsymbol{j}_1 &= \boldsymbol{j} \\ \boldsymbol{k}_1 &= \cos\alpha \boldsymbol{k} + \sin\alpha \boldsymbol{i} \end{aligned}$$

  and between $x_1 y_1 z_1$ and $x_B y_B z_B$ is:

$$\begin{aligned} \boldsymbol{i}_B &= \cos\varphi \boldsymbol{i}_1 + \sin\varphi \boldsymbol{j}_1 \\ \boldsymbol{j}_B &= \cos\varphi \boldsymbol{j}_1 - \sin\varphi \boldsymbol{i}_1 \\ \boldsymbol{k}_B &= \boldsymbol{k}_1 \end{aligned}$$

  The angular velocity of the fuselage is $\boldsymbol{\omega}_f = \dot{\varphi} \boldsymbol{k}_1 = \dot{\varphi} \boldsymbol{k}_B$. Then in the inertial reference frame $\boldsymbol{\omega}_f = \dot{\varphi}(\cos\alpha \boldsymbol{k} + \sin\alpha \boldsymbol{i})$.

  The angular velocity of the main rotor is: $\boldsymbol{\omega}_1 = \boldsymbol{\omega}_f - \dot{\phi}_1 \boldsymbol{k}_B = (\dot{\varphi} - \dot{\phi}_1)\sin\alpha \boldsymbol{i} + (\dot{\varphi} - \dot{\phi}_1)\cos\alpha \boldsymbol{k}$. For the case of the tail rotor: $\boldsymbol{\omega}_2 = \boldsymbol{\omega}_f + \dot{\phi}_2 \boldsymbol{j}_B$. Therefore referring to the inertial reference frame: $\boldsymbol{\omega}_2 = (\dot{\phi}_2 \sin\varphi\cos\alpha + \dot{\varphi}\sin\alpha)\boldsymbol{i} + \dot{\phi}_2 \cos\varphi \boldsymbol{j} + (\dot{\varphi}\cos\alpha + \dot{\phi}_2 \sin\varphi \sin\alpha)\boldsymbol{k}$

- Compute the absolute velocities (with respect to an inertial reference frame) of the centers of mass G_1 and G_2 of the two rotors, respectively, as a function of the absolute velocity of the center of mass of the whole system ($\boldsymbol{v}_G$), $\alpha, \varphi, \dot{\theta}$ and $\dot{\varphi}$.

  We can obtain the position vector as $\boldsymbol{r}_{GG_1} = \boldsymbol{r}_{GG_f} + \boldsymbol{r}_{G_f G_1}$ and the angular velocity is $\boldsymbol{\omega}_f = \dot{\varphi}(\cos\alpha \boldsymbol{k} + \sin\alpha \boldsymbol{i})$. For the second rotor $\boldsymbol{r}_{GG_1} = \boldsymbol{r}_{GG_f} + \boldsymbol{r}_{G_f G_1}$. Then:

$$\begin{aligned} \boldsymbol{v}_{G_1} &= \boldsymbol{v}_G + \omega_{\mathbf{f}} \times \boldsymbol{r}_{\mathbf{GG_1}} \\ \boldsymbol{v}_{G_2} &= \boldsymbol{v}_G + \omega_{\mathbf{f}} \times \boldsymbol{r}_{\mathbf{GG_2}} \end{aligned}$$

- Compute the kinetic energy, T, of the whole system. Consider that the helicopter is able to move horizontally, with x_G and y_G being the position of the center of mass with respect to the inertial reference frame. The kinetic energy of the whole system is the sumation on the contributions of fuselage and rotors:

$T = T_f + T_{R1} + T_{R2}$. The kinetic energy of the spacecraft and rotors is computed as the kinetic energy of a rigid body. The absolute velocity of the center of mass of the system is expressed as $\dot{x_G}\boldsymbol{i} + \dot{y_G}\boldsymbol{j} + \dot{z_G}\boldsymbol{k}$.

Note that the inertia tensor of each rotor is obtained by replacing the lengths, anf the one of the fuselage is:

$$\bar{\bar{I}}_{G_f}^{fus} = \frac{1}{12}ML^2 \begin{bmatrix} \frac{1}{50} & 0 & 0 \\ 0 & \frac{101}{100} & 0 \\ 0 & 0 & \frac{101}{100} \end{bmatrix}$$

Then:

$$\begin{aligned} T_{\mathrm{f}} &= \frac{1}{2}m_f|\mathbf{v}_G|^2 + \frac{1}{2}\boldsymbol{\omega}_f\bar{\bar{I}}_G\boldsymbol{\omega}_f \\ T_{R_1} &= \frac{1}{2}m_{R_1}|\mathbf{v}_{G_1}|^2 + \frac{1}{2}\boldsymbol{\omega}_1\bar{\bar{I}}_{G_1}\boldsymbol{\omega}_1 \\ T_{R_2} &= \frac{1}{2}m_{R_2}|\mathbf{v}_{G_2}|^2 + \frac{1}{2}\boldsymbol{\omega}_2\bar{\bar{I}}_{G_2}\boldsymbol{\omega}_2 \end{aligned}$$

Please note that the angular velocity should be expressed in the same reference frame as the inertia tensor to perform the computation, in these cases in the fuselage-fixed and rotor-fixed reference frames.

- Compute the angular momentum of each of the rotor with respect to its center of mass (G_1 and G_2)

  The angular momentum of the rotors with respect to their center of mass can be computed as:

$$\begin{aligned} \mathbf{H}_{G_1}^{R_1} &= \bar{\bar{I}}_{G_1}\boldsymbol{\omega}_1 \\ \mathbf{H}_{G_2}^{R_2} &= \bar{\bar{I}}_{G_2}\boldsymbol{\omega}_2 \end{aligned}$$

  Once again, the angular velocity should be expressed in the same reference frame as the inertia tensor to perform the computation.

- Compute the angular momentum of the whole system with respect to the center of mass of the whole system.

  The angular momentum of the whole system can be computed as the contribution of the fuselage and the rotors: $\mathbf{H}_G = \mathbf{H}_G^{\mathrm{f}} + \mathbf{H}_G^{R_1} + \mathbf{H}_G^{R_2}$. The angular momentum of the fuselage with respect to the fuselage-fixed frame is:

$$\mathbf{H}_{G_f}^{\mathrm{f}} = \bar{\bar{I}}_G^{\mathrm{fus}}\boldsymbol{\omega}_f \tag{34}$$

  and then, the angular momentum of the fuselage with respect to the center of mass of the whole system, is:

$$\mathbf{H}_G^{\mathrm{f}} = \mathbf{H}_{G_f}^{\mathrm{f}} + m_f\mathbf{GG}_f \times \mathbf{v}_{G_f} \tag{35}$$

  where $\boldsymbol{v}_{G_f} = \boldsymbol{v}_G + \omega_{\mathbf{f}} \times \boldsymbol{r}_{\mathbf{GG_1}}$.

  Following the same approach for computing the angular momentum of the rotor 1 with respect to the center of mass of the whole system, we use:

$$\mathbf{H}_G^{R_1} = \mathbf{H}_{G_1}^{R_1} + m_{R_1}\mathbf{GG}_1 \times \mathbf{v}_{G_1} \tag{36}$$

  where $\mathbf{GG}_1$ is given in question 3.

  The angular momentum of the tail rotor with respect to the center of mass of the whole system can be computed following the same procedure.

- Obtain the set of ordinary differential equations that are required in order to obtain the position of the rotorcraft and φ, ϕ_1, ϕ_2 as a function of time.

  Second Newton's law establishes:

$$m_t\frac{\mathrm{d}\boldsymbol{v}_G}{\mathrm{d}t} = \boldsymbol{F}_{R_1} + \boldsymbol{F}_{R_2} + m_t g\boldsymbol{k} \tag{37}$$

This is a vector equation and thus it has 3 components. To compute the moment equation we need first to obtain the moment of inertia in the center of gravity of the system. Steiner's theorem establishes:

$$\bar{I_{G_i}} = \bar{I_G} + m\left(r_G^2\bar{U} - [\boldsymbol{r}_G\boldsymbol{r}_G]\right) \tag{38}$$

With this equation it is possible to compute the I_G for each of the three rigid bodies. The total moment of inertia is the sum of the three terms. Then the moment equation is:

$$I_G\boldsymbol{\alpha} = \sum \boldsymbol{M}_G = \boldsymbol{M}_{R_1} + \boldsymbol{GG_{R_1}} \times \boldsymbol{F}_{R_1} + \boldsymbol{GG_{R_2}} \times \boldsymbol{F}_{R_2} \tag{39}$$