Please answer the questions 1-10 with True or False and provide a short explanation for your answer. **A right answer without explanation will not be considered.** .

1. Two birds are sitting at the top of a tree at the same height. There is no wind. The mass of the first bird, m_1, is three times larger than the mass of the second bird, m_2. The aerodynamic efficiency $E_1 = L/D$ of the first bird is 10% larger than the aerodynamic efficiency of the second bird, E_2 (Recall that L is the lift and D is the drag). Both birds start gliding at a constant speed towards the forest floor and it is possible to assume than in both cases the gliding angle is very small. The second bird being much lighter than the first bird will land farther than the first bird. (True/False)

   False. The gliding angle depends only on the aerodynamic efficiency. The larger the efficiency the smaller the angle. In this case, bird 1 will land farther.

2. If a particle moves along a circular path, then the tangential acceleration of the particle is always zero. (True/False)

   False. If the particle is moving at a variable speed then the tangential acceleration will not be zero.

3. If a moving reference frame is translating without rotating with respect to a fixed reference frame, then we can say that the moving reference frame is an inertial reference frame. (True/False)

   False. The translation of the moving reference frame must be at a constant speed.

4. In relative motion, the vector $\boldsymbol{\omega} \times \boldsymbol{r}'_P$ that appears in the expression of the velocity of point P is not necessarily parallel to the vector $\boldsymbol{\alpha} \times \boldsymbol{r}'_P$ that appears in the expression of the acceleration of point P. Recall that $\boldsymbol{\omega}$ and $\boldsymbol{\alpha}$ are the angular velocity and angular acceleration, respectively, of the moving reference frame with respect to the fixed reference frame and $\boldsymbol{r}'_P$ is the relative position of P with respect to the moving reference frame. (True/False)

   True. In general, $\boldsymbol{\omega}$ and $\boldsymbol{\alpha}$ do not need to be parallel, although in some situations they may be parallel.

5. Consider a damped harmonic oscillator of natural frequency ω_0 with small damping parameter γ under the influence of an external periodic force of frequency ω_1. For long times, no matter what initial conditions we choose, the oscillations are ultimately governed solely by the external force. Therefore, the amplitude of the resulting oscillations is independent of ω_0 and depends only on the amplitude of the oscillations of the external periodic force. (True/False)

   False. The amplitude of the resulting oscillations do depend on ω_0. For example, if $\omega_1 \simeq \omega_0$ resonance occurs and the resulting oscillations might be very large.

6. Consider the motion of a point particle of mass m with respect to an inertial reference frame under the influence of a force

$$\boldsymbol{F} = g(r)\boldsymbol{r},$$

   where $\boldsymbol{r}$ is the position vector of the particle and $g(r)$ is a known scalar function of r. In such a situation, the angular momentum of the particle with respect to the origin is conserved and using this fact it is possible to show that the trajectory of the point particle is planar. (True/False)

   True. The moment of the force is zero. Then, the angular momentum vector is constant. Since the angular momentum is perpendicular to the position vector and to the velocity vector, then the position and the velocity are always contained in a plane. This plane does not change since the angular momentum is constant. Therefore the trajectory is planar.

7. A rigid body is in pure rotation. Two points of the body (A and B) are located at the same distance to the instantaneous axis of rotation. Then, the vector velocity of both points is the same, $\boldsymbol{v}_A = \boldsymbol{v}_B$. (True/False)

   False. The magnitude is the same but the direction of the vector might be different.

8. Consider a rigid body of arbitrary shape. The moment of inertia tensor with respect to the center of mass G does not depend on the orientation of the body axes as long as the origin of the body axes is G. (True/false)

   False. By definition, the moment of inertia tensor depends on the orientation of the axis.

9. Consider a system of n_1 rigid bodies and n_2 point particles mutually interacting. There are no external forces or couples acting on the system. Then, the center of mass of the system is at rest or moves at constant speed along a straight line. (True/False)

True. Since there are no external forces, the resultant force acting on the system is zero and the acceleration of the centre of mass of the system is zero as well. Then, the velocity of the centre of mass is constant. If this constant is zero then the centre of mass of the system is at rest. If the constant is different than zero then the centre of mass of the system moves at constant speed along a straight line.

10. A rigid body is in general motion under a system of forces. The work done by the resultant moment about the centre of mass has the effect of increasing the rotational kinetic energy and not the traslational kinetic energy. (True/False)

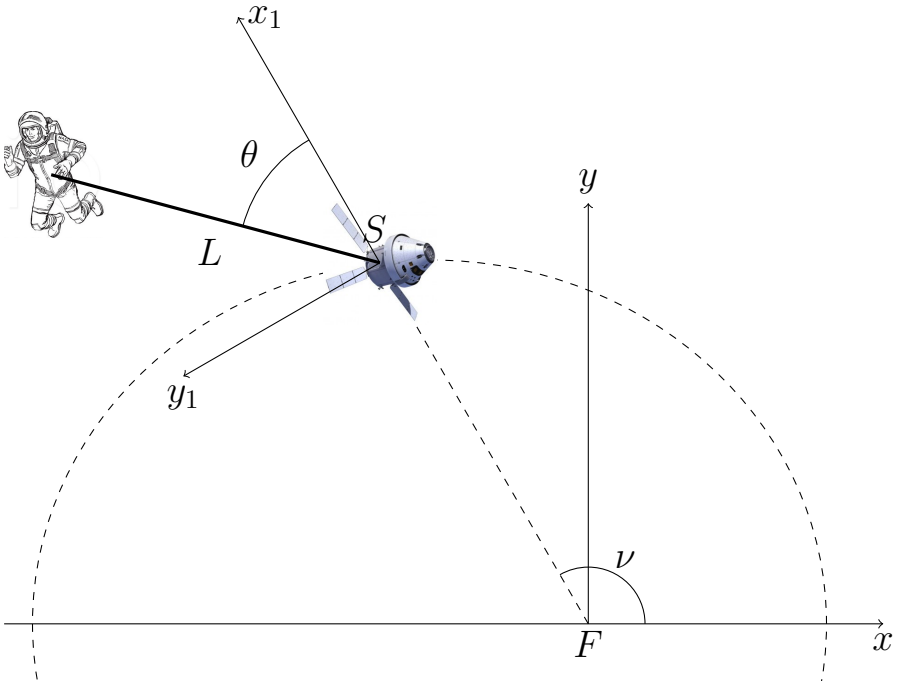
    True. This is a consequence of the fact that the translational motion of the body can be studied by reducing the body to the centre of mass. Then, the work done by the resultant force has the effect of increasing the traslational kinetic energy only. By combining that fact, with the global balance of the rigid body, it is possible to show that the work done by the resultant moment about the centre of mass has the effect of increasing the rotational kinetic energy only.

Problem 1

A spacecraft S is orbiting a celestial body F. The trajectory of the spacecraft with respect to an inertial reference frame Fxy is given by

$$r = \frac{p}{1 + e\cos\nu},$$

where p and e are known constants and $\nu(t)$ is a known function of time. The space explorer Ijon Tichy needs to perform some extra-vehicular activities during his Seventh Voyage[1]. He attaches himself with a massless cord of length L to the spacecraft and goes for a space walk. We neglect all gravitational interactions and assume that we can consider him as a point particle of mass m. We define a moving reference frame Sx_1y_1 as indicated in the figure.



1. Determine the acceleration of the spacecraft with respect to Fxy as a function of time. (**2 points**)

   Using polar coordinates, noting that vector $\boldsymbol{i}_1$ coincides with the radial vector and the vector $\boldsymbol{j}_1$ coincides with the azimuthal vector then the velocity and acceleration of S are

   $$\boldsymbol{v}_S = \dot{r}\boldsymbol{i}_1 + r\dot{\nu}\boldsymbol{j}_1, \tag{1}$$
   $$\boldsymbol{a}_S = (\ddot{r} - r\dot{\nu}^2)\boldsymbol{i}_1 + (r\ddot{\nu} + 2\dot{r}\dot{\nu})\boldsymbol{j}_1, \tag{2}$$

   with

   $$r = \frac{p}{1 + e\cos\nu}, \tag{3}$$
   $$\dot{r} = \frac{pe\dot{\nu}\sin\nu}{(1 + e\cos\nu)^2}, \tag{4}$$
   $$\ddot{r} = \frac{pe}{(1 + e\cos\nu)^2}\left[\ddot{\nu}\sin\nu + \dot{\nu}^2\left(\cos\nu + \frac{2e\sin^2\nu}{1 + e\cos\nu}\right)\right]. \tag{5}$$

   Since p, e and $\nu(t)$ are known, then all terms in eqns. (3)-(5) are known. Substituting these formulas in eqns. (1)-(2) we obtain the velocity and acceleration of S as a function of time. In particular, in the following we will use

   $$\boldsymbol{a}_S = A_1\boldsymbol{i}_1 + A_2\boldsymbol{j}_1, \tag{6}$$

   where A_1 and A_2 are known and given by

   $$A_1(t) = \frac{pe}{(1 + e\cos\nu)^2}\left[\ddot{\nu}\sin\nu + \dot{\nu}^2\left(\cos\nu + \frac{2e\sin^2\nu}{1 + e\cos\nu}\right)\right] - \dot{\nu}^2\frac{p}{1 + e\cos\nu}, \tag{7}$$
   $$A_2(t) = \ddot{\nu}\frac{p}{1 + e\cos\nu} + \dot{\nu}^2\frac{2pe\sin\nu}{(1 + e\cos\nu)^2}. \tag{8}$$

[1] *The Star Diaries* by Stanislaw Lem.

2. Determine the tangent vector to the trajectory of S as a function of time. (**1 point**)

   The tangent vector is given by

$$\boldsymbol{e}_t = \frac{\boldsymbol{v}_S}{v_S} = \frac{1}{\sqrt{\dot{r}^2 + r^2\dot{\nu}^2}}(\dot{r}\boldsymbol{i}_1 + r\dot{\nu}\boldsymbol{j}_1), \tag{9}$$

   where it is necessary to substitute the values of r and $\dot{r}$ from eqns. (3) and (4).

3. Determine the relative velocity and relative acceleration of Ijon Tichy with respect to Sx_1y_1 as a function of θ and its time derivatives. (**2 points**)

   Now for the analysis of the motion of Ijon Tichy, we can define again polar coordinates with respect to Sx_1y_1. The radial vector $\boldsymbol{e}_r$ is the unit vector in the direction connecting the spacecraft to Ijon Tichy and the azimuthal vector $\boldsymbol{e}_\theta$ is perpendicular to it. Then the relative position, relative velocity and relative acceleration are given by

$$\boldsymbol{r}'_{IT} = L\boldsymbol{e}_r, \tag{10}$$
$$\boldsymbol{v}'_{IT} = L\dot{\theta}\boldsymbol{e}_\theta, \tag{11}$$
$$\boldsymbol{a}'_{IT} = -L\dot{\theta}^2\boldsymbol{e}_r + L\ddot{\theta}\boldsymbol{e}_\theta. \tag{12}$$

4. Determine the fictitious forces acting on Ijon Tichy if we analyze the problem with respect to the frame Sx_1y_1. (**3 points**)

   The fictitious forces are given by the sum of four terms

$$\boldsymbol{F}_I = \boldsymbol{F}_1 + \boldsymbol{F}_2 + \boldsymbol{F}_3 + \boldsymbol{F}_4 = -m\left(\boldsymbol{a}_S + \boldsymbol{\alpha}\times\boldsymbol{r}'_{IT} + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r}'_{IT}) + 2\boldsymbol{\omega}\times\boldsymbol{v}'_{IT}\right) \tag{13}$$

   Using eq. (6) we obtain the first term

$$\boldsymbol{F}_1 = -m(A_1\boldsymbol{i}_1 + A_2\boldsymbol{j}_1) = -m(A_1\cos\theta + A_2\sin\theta)\boldsymbol{e}_r + m(A_1\sin\theta - A_2\cos\theta)\boldsymbol{e}_\theta. \tag{14}$$

   In order to obtain the remaining three terms we need the angular velocity and acceleration of Sx_1y_1 with respecto to Fxy. These are known and given by

$$\boldsymbol{\omega} = \dot{\nu}\boldsymbol{k}, \boldsymbol{\omega} = \ddot{\nu}\boldsymbol{k}. \tag{15}$$

   Therefore, we obtain

$$\boldsymbol{F}_2 = -mL\ddot{\nu}\boldsymbol{e}_\theta, \tag{16}$$
$$\boldsymbol{F}_3 = mL\dot{\nu}^2\boldsymbol{e}_r, \tag{17}$$
$$\boldsymbol{F}_4 = 2mL\dot{\nu}\dot{\theta}\boldsymbol{e}_r. \tag{18}$$

5. Write down one ordinary differential equation that once solved (with given initial conditions) would yield the angle θ as a function of time. (You do not need to solve it) (**1 point**)

   The only external force acting on Ijon Tichy is the tension from the cord, $\boldsymbol{F} = -T\boldsymbol{e}_r$. Therefore, the desired equation is Newton's second law $\boldsymbol{F} + \boldsymbol{F}_I = m\boldsymbol{a}'_{IT}$ along $\boldsymbol{e}_\theta$.

$$A_1\sin\theta - A_2\cos\theta - L\ddot{\nu} = L\ddot{\theta}. \tag{19}$$

6. Assuming that you have solved the previous equation, and therefore you know $\theta(t)$, determine the value of the tension on the cord as a function of time. (**1 point**)

   This is obtained from Newton's second law along $\boldsymbol{e}_r$

$$-T - m(A_1\cos\theta + A_2\sin\theta) + mL\dot{\nu}^2 + 2mL\dot{\nu}\dot{\theta} = -mL\dot{\theta}^2. \tag{20}$$

Problem 2

Space tethers are cables used in space for propulsion, stabilization or other purposes. The space tether is modeled as a dumbbell, that is, as a homogeneous rigid rod of mass $m_T = 2M$ and length L, with two point particles at both ends with masses $m_1 = M$ and $m_2 = M$. The motion of a space tether is to be analyzed in this exercise using an inertial reference frame $Oxyz$ and a reference frame attached to the system $Gx_By_Bz_B$ where initially $G = O$, as shown in the figure. This reference frame is defined with origin at the center of mass of the system, G, with the axis Gx_B along the tether in the direction from m_2 to m_1. Gy_B is defined as the direction perpendicular to both Gx_B and Gz as it can be seen in the figure. Finally, Gz_B is perpendicular to Gx_B and Gy_B. The orientation of the tether is determined by two angles: φ the angle between the projection of the tether in the plane Oxy and Gx_B, and θ the angle between Ox and that projection, as it can be seen in the figure. In addition, the rigid rod is considered one-dimensional and its tensor of inertia with respect to its center of mass in the body fixed reference frame $Gx_By_Bz_B$ is equal to:

$$\bar{\bar{I}}_G = \frac{1}{12} m_T L^2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

It is considered that the tether is in deep space, so all the external forces and torques are negligible.

- Compute the angular velocity of the tether $\boldsymbol{\omega}$ and the absolute velocities of the particles m_1 and m_2 as a function of the absolute velocity of the center of mass of the whole system, $\theta, \varphi, \dot{\theta}$ and $\dot{\varphi}$. (**2 points**)

  The angular velocity has two terms: $\boldsymbol{\omega} = \dot{\theta}\mathbf{k} - \dot{\varphi}\mathbf{j}_B$. In body-fixed axis (given that $\mathbf{k} = \sin\varphi\mathbf{i}_B + \cos\varphi\mathbf{k}_B$) the resulting angular velocity of the tether system turns out to be:

  $$\boldsymbol{\omega} = \dot{\theta}\sin\varphi\mathbf{i}_B - \dot{\varphi}\mathbf{j}_B + \dot{\theta}\cos\varphi\mathbf{k}_B$$

  The end masses are attached to the rod. Therefore, they can be considered as particles of the rigid body and the velocity field can be used to compute their velocity:

  $$\begin{aligned} \mathbf{v}_1 &= \mathbf{v}_G + \boldsymbol{\omega}\times\mathbf{r}_1 = \mathbf{v}_G + \frac{L}{2}\dot{\theta}\cos\varphi\mathbf{j}_B + \frac{L}{2}\dot{\varphi}\mathbf{k}_B \\ \mathbf{v}_2 &= \mathbf{v}_G + \boldsymbol{\omega}\times\mathbf{r}_2 = \mathbf{v}_G - \frac{L}{2}\dot{\theta}\cos\varphi\mathbf{j}_B - \frac{L}{2}\dot{\varphi}\mathbf{k}_B \end{aligned}$$

  where $\mathbf{r}_1 = L/2\mathbf{i}_B$, $\mathbf{r}_2 = -L/2\mathbf{i}_B$ and $\mathbf{v}_G$ is the velocity of the center of mass.

- Compute the kinetic energy, T, of the whole system (**2 points**)

  The kinetic energy of the whole system has the contribution of rod and masses: $T = T_{rod} + T_{m_1} + T_{m_2}$. The kinetic energy of the rod can be computed as the kinetic energy of a rigid body whereas the contribution of the masses is computed as point particles.

  $$\begin{aligned} T_{\text{rod}} &= \frac{1}{2}m_T|\mathbf{v}_G|^2 + \frac{1}{2}\boldsymbol{\omega}\bar{\bar{I}}_G\boldsymbol{\omega} = M|\mathbf{v}_G|^2 + \frac{1}{12}ML^2\left(\dot{\varphi}^2 + \dot{\theta}^2\cos^2\varphi\right) \\ T_{m_1} &= \frac{1}{2}M\left(\dot{\xi}^2 + \left(\dot{\eta}^2 + \frac{L}{2}\dot{\theta}\cos\varphi\right)^2 + \left(\dot{\zeta}^2 + \frac{L}{2}\dot{\varphi}\right)^2\right) \\ T_{m_2} &= \frac{1}{2}M\left(\dot{\xi}^2 + \left(\dot{\eta}^2 - \frac{L}{2}\dot{\theta}\cos\varphi\right)^2 + \left(\dot{\zeta}^2 - \frac{L}{2}\dot{\varphi}\right)^2\right) \end{aligned}$$

  where $\mathbf{v}_G = \dot{\xi}\mathbf{i}_B + \dot{\eta}\mathbf{j}_B + \dot{\zeta}\mathbf{k}_B$. Please note that the angular velocity should be expressed in the same reference frame as the inertia tensor to perform the computation. The kinetic energy of the whole system is then:

  $$T = 2M|\mathbf{v}_G|^2 + \frac{1}{3}ML^2\left(\dot{\varphi}^2 + \dot{\theta}^2\cos^2\varphi\right)$$

- Compute the angular momentum of the whole system with respect to the center of mass of the whole system (**2 points)**

In the same way as in previous section, the angular momentum of the whole system can be computed as the contribution of the rod and the end masses: $\mathbf{H}_G = \mathbf{H}_{G,\text{rod}} + \mathbf{H}_{G,m_1} + \mathbf{H}_{G,m_2}$

$$\begin{aligned}
\mathbf{H}_{G,\text{rod}} &= \bar{\bar{I}}_G\boldsymbol{\omega} = \frac{1}{6}ML^2\left(-\dot{\varphi}\mathbf{j}_B + \dot{\theta}\cos\varphi\mathbf{k}_B\right)\\
\mathbf{H}_{G,m_1} &= m_1\mathbf{r}_1\times\mathbf{v}_1 = \frac{1}{2}ML\left[\left(\dot{\eta}+\frac{L}{2}\dot{\theta}\cos\varphi\right)\mathbf{k}_B - \left(\dot{\zeta}+\frac{L}{2}\dot{\varphi}\right)\mathbf{j}_B\right]\\
\mathbf{H}_{G,m_2} &= m_2\mathbf{r}_2\times\mathbf{v}_2 = -\frac{1}{2}ML\left[\left(\dot{\eta}-\frac{L}{2}\dot{\theta}\cos\varphi\right)\mathbf{k}_B - \left(\dot{\zeta}-\frac{L}{2}\dot{\varphi}\right)\mathbf{j}_B\right]
\end{aligned}$$

Thus, the angular momentum of the whole system with respect to the center of mass turns out to be:

$$\mathbf{H}_G = \frac{2}{3}ML^2\left(-\dot{\varphi}\mathbf{j}_B + \dot{\theta}\cos\varphi\mathbf{k}_B\right)$$

- Initially, the tether has the following conditions:

$$\mathbf{r}_G(t=0)=\mathbf{0}\quad \mathbf{v}_G(t=0)=\mathbf{0}\quad \theta(t=0)=0\quad \varphi(t=0)=\pi/4\quad \omega(t=0)=\frac{1}{\sqrt{2}}\left[\frac{1}{\sqrt{2}}\mathbf{i}_B+\mathbf{j}_B+\frac{1}{\sqrt{2}}\mathbf{k}_B\right]$$

Compute the evolution of θ and φ as a function of time (**4 points**)

From the given initial angular velocity, the initial conditions in $\dot{\varphi}$ and $\dot{\theta}$ are: $-\dot{\varphi}_0 = \dot{\theta}_0 = 1/\sqrt{2}$.
There are no external forces or moments acting on the system, so the equations of motion read:

$$\begin{aligned}
4M\mathbf{a}_G &= 0\\
\frac{\mathrm{d}\mathbf{H}_G}{\mathrm{d}t} &= \left.\frac{\mathrm{d}\mathbf{H}_G}{\mathrm{d}t}\right|_B + \boldsymbol{\omega}\times\mathbf{H}_G = 0
\end{aligned}$$

From the first vector equation, we get that $\mathbf{v}_G$ is constant, and using the initial conditions for velocity and position of the center of mass, we conclude that G is always coincident with O. Therefore, $\dot{\xi}=\dot{\eta}=\dot{\zeta}=0$.
From the second vector equation, we get second order ordinary differential equations:

$$\ddot{\varphi} + \dot{\theta}^2\sin\varphi\cos\varphi = 0 \tag{21}$$
$$\ddot{\theta}\cos\varphi - 2\dot{\theta}\dot{\varphi}\sin\varphi = 0 \tag{22}$$

Apart from the equations of motion, the energy equation can be used because the conservation of energy holds. The potential energy is zero because there are no external forces. The initial value of the kinetic energy is $T_0 = ML^2/4$. Therefore:

$$\dot{\varphi}^2 + \dot{\theta}^2\cos^2\varphi = \frac{3}{4} \tag{23}$$

The problem is well posed and closed with equations (21-23). Nevertheless, it is not easy to get to an analytical solution using the previous set of equations. Instead, the fact that the angular momentum vector is constant can be used to provide additional equations. Given that the angular momentum vector is constant, the projection of the vector in any inertial direction is also constant. In particular, the component of $\mathbf{H}_G$ in $\mathbf{k}$ direction provide us with the following condition:

$$\mathbf{H}_G\cdot\mathbf{k} = \dot{\theta}\cos^2\varphi = \frac{1}{2\sqrt{2}} \tag{24}$$

Therefore, the evolution of φ is reduced to a quadrature:

$$t = \pm\int\frac{\mathrm{d}\varphi}{\sqrt{\frac{3}{4}-\frac{1}{8\cos^2\varphi}}}$$

Knowing $\varphi(t)$, equation (24) gives the quadrature for θ:

$$2\sqrt{2}\int \mathrm{d}\theta = \int \frac{\mathrm{d}t}{\cos^2\varphi}$$

- Assuming that it is possible to extend or contract the tether at will, so that the tether has a variable length, compute the required length of the tether, ℓ_F, to reduce the component of the angular velocity in the $\mathbf{j_B}$ direction to half of its initial value $\omega_F \cdot \mathbf{j_B} = \frac{1}{2\sqrt{2}}$. Show whether the resulting angular velocity is also reduced by half. (**1 extra point**)

If the energy remains constant, the equation of energy will give us a relationship between the initial and the required length of the tether.

$$\left(\frac{\ell_F}{L}\right)^2 = \frac{\frac{3}{4}}{\dot{\varphi}^2 + \dot{\theta}^2\cos^2\varphi}$$

The condition $\omega_F \cdot \mathbf{j_B} = \frac{1}{2\sqrt{2}}$ is equivalent to $\dot{\varphi}_F^2 = 1/8$. Besides, the solution of the previous section for θ is still valid. Substituting, we obtain

$$\left(\frac{\ell_F}{L}\right)^2 = \frac{6\cos^2\varphi}{1+\cos^2\varphi}$$

So the tether length would depend on the evolution of φ.

On the other hand, given that $\dot{\varphi}_F^2 = 1/8$ and $\dot{\theta}_F^2 = 1/(8\cos^2\varphi)$, the magnitude of the angular velocity at that instant will be a function of φ. Initially, $|\boldsymbol{\omega}_0| = \dot{\varphi}_0^2 + \dot{\theta}_0^2 = 1$. Therefore, the magnitude of the angular velocity is not reduced by half.