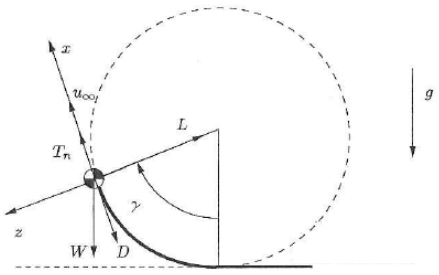
Please answer the questions 1-6 with True or False and provide a short explanation for your answer. **A right answer without explanation will not be considered.** Answer also exercise 7.

1. In a flight exhibition, an aircraft is performing a circular loop of radius R at constant speed, as indicated in the figure below. The forces acting on the aircraft are the aerodynamic lift L and drag D, the weight W and the thrust T_n. The manufacturer of the aircraft says that the load factor should not exceed $n = L/W = 5$. Therefore, the maximum velocity at which the loop can be performed is $u_{\infty,max} = 2\sqrt{gR}$, where g is the gravitational acceleration. (True/False)



True. Newton's second law in normal direction yields

$$L - W\cos\gamma = ma_n = m\frac{u_\infty^2}{R}. \tag{1}$$

The load factor is given by

$$n = m\frac{u_\infty^2}{WR} + \cos\gamma = \frac{u_\infty^2}{gR} + \cos\gamma, \tag{2}$$

which is maximum when $\gamma = 0$. Introducing $n_{max} = 5$ and solving for u_∞ we obtain $u_{\infty,max} = 2\sqrt{gR}$.

2. Consider the motion of a point particle in 3D. If we know the arc length coordinate as a function of time, $s(t)$, the radius of curvature as a function of time, $\rho(t)$, and the initial position of the particle, $\boldsymbol{r}_0$, then it is possible to determine for all times the position of the particle as a function of time, $\boldsymbol{r}(t)$. (True/False)

   False. We can decompose the determination of $\boldsymbol{r}(t)$ in two parts: $\boldsymbol{r}(s)$ and $s(t)$. Obviously we know the second part, but we do not have enough information to determine $\boldsymbol{r}(s)$. For example, at $t = 0$ we do not know the direction of the tangent vector, since we do not know the initial velocity vector of the particle.

3. Consider the motion of a point particle P with respect to a moving reference frame $O_1x'y'z'$ and with respect to an inertial reference frame $Oxyz$.

   (a) If the motion of the reference frame $O_1x'y'z'$ with respect to the inertial reference frame $Oxyz$ corresponds to a translation without rotation, then the relative acceleration of the particle $\boldsymbol{a}'_P$ will be the same as the absolute acceleration $\boldsymbol{a}_P$. (True/False)

   False. We do not know how the point O_1 is translating with respect to $Oxyz$. In general, it will be moving with a velocity and an acceleration. Therefore the absolute velocity is given by:

$$\boldsymbol{a}_P = \boldsymbol{a}'_P + \boldsymbol{a}_{O_1}. \tag{3}$$

   Only if O_1 moves at a constant velocity the proposed statement would be true.

   (b) If the motion of the reference frame $O_1x'y'z'$ with respect to the inertial reference frame $Oxyz$ corresponds to a rotation without translation, then the projection of the relative velocity $\boldsymbol{v}'_P$ along the direction of the angular velocity vector $\boldsymbol{\omega}$ will be equal to the projection of the absolute velocity $\boldsymbol{v}_P$ along that direction, where the angular velocity of $O_1x'y'z'$ with respect to $Oxyz$ is $\boldsymbol{\omega}$. (True/False)

   True. If there is no tranlation of $O_1x'y'z'$ then $\boldsymbol{v}_{O_1} = 0$. The absolute velocity is given by

$$\boldsymbol{v}_P = \boldsymbol{v}'_P + \boldsymbol{\omega} \times \boldsymbol{r}'_P. \tag{4}$$

   Scalar product of the previous equation with $\boldsymbol{\omega}$ leads to $\boldsymbol{\omega} \cdot \boldsymbol{v}_P = \boldsymbol{\omega} \cdot \boldsymbol{v}'_P$, as stated above.

4. Consider a point particle P of mass m which is allowed to move in 1D only. It is attached to a wall with a spring of constant k and a damper of constant λ. We apply an external periodic force of amplitude $F_1 > 0$ and with frequency ω_1.

(a) If m, λ and k are fixed, resonance will occur at a higher frequency the larger the value of the amplitude F_1. (True/False)

False. Resonance will occur when $\omega_1 = \omega_0 = \sqrt{k/m}$ independently of the amplitude F_1.

(b) If m, λ and k are fixed, and $\omega_1^2 = k/m$, for long times the amplitude of the resulting oscillations of P will increase with time. (True/False)

False. For long times, the influence of the initial conditions has vanished, and then the solution is periodic with frequency equal to the forcing frequency ω_1.

(c) If m and k are fixed, $\omega_1^2 = k/m$, the smaller the value of λ the larger the amplitude of the resulting oscillations of P. (True/False)

True. At resonance the amplitude of the oscillations is proportional to $1/\lambda$ (see eq. 4.63 of the lecture notes) and therefore the statement is true.

5. Consider the motion of the Earth and the Sun (ignoring all other planets and celestial objects). The only force acting in the system is the mutual gravitational attraction. Since the mass of the Sun is much larger than the mass of the Earth we can assume that the centre of mass of the system is in the Sun. Therefore, we can consider the problem of the motion of the Earth as a Kepler problem in which the Sun is at the origin and the Earth moves around it under the influence of the gravitational attraction from the Sun.

(a) Using conservation of angular momentum it is possible to show that the orbit of the Earth is contained in a plane parallel to the angular momentum vector. (True/False)

False. The angular momentum vector $\boldsymbol{h}$ is defined as $\boldsymbol{h} = m\boldsymbol{r} \times \boldsymbol{v}$, therefore it is perpendicular to the position and the velocity of the particle. So if the angular momentum is constant, the orbit is contained in the plane perpendicular (not parallel) to the orbit of the Earth.

(b) The speed of the Earth in its motion around the Sun is constant. (True/False)

False. This statement is, in general, false. The solution of the Kepler problem leads to a conic section, which in the case of the Earth moving around the Sun is an ellipse, with the Sun in one of the foci. According to the Kepler's second law: "The line joining a planet to the Sun sweeps out equal areas in equal intervals of time", and therefore, describing the ellipse the speed of the Earth is not constant. One could argue that if we do not know the initial conditions of the problem, it could be that with some particular initial conditions the orbit of the Earth yields a circle. In that case, the statement would be true.

6. Consider a rigid body with a body-fixed reference attached to it $Gx_By_Bz_B$ which is moving (in general motion: translation and rotation) with respect to an inertial reference frame $Oxyz$. The motion of the body is such that the axis Gx_B is always contained within the plane Oxy and rotates at a constant rate Ω_1 with respect to the axis Oz. The axis Gz_B is always parallel to the plane perpendicular to Gx_B and rotates at a constant rate Ω_2 with respect to the axis Gx_B. The axis Gy_B is always perpendicular to Gx_B and Gz_B, so as to define a right-handed coordinate system.

(a) It is possible to determine the angular velocity vector in spite of the fact that we do not know how the point G is moving. (True/False)

True. It is indeed possible since the motion of G would give us the translation of the body and not its rotation. The angular velocity is given by

$$\boldsymbol{\omega} = \Omega_1 \boldsymbol{k} + \Omega_2 \boldsymbol{i_B} = \Omega_1 \boldsymbol{k} + \Omega_2(\cos(\Omega_1 t)\boldsymbol{i} + \sin(\Omega_1 t)\boldsymbol{j}) \tag{5}$$

(b) Besides, the angular acceleration of the body is zero, $\boldsymbol{\alpha} = \boldsymbol{0}$. (True/False)

False. Computing the time derivative of eq. (5) it is clear that $\boldsymbol{\alpha} \neq \boldsymbol{0}$

(c) For the motion described above, in order to define the instantaneous axis of rotation, we need to identify the point P with zero velocity. If we know the position and velocity of G, this can be computed from the following formula

$$\boldsymbol{GP} = \frac{\boldsymbol{\omega} \times \boldsymbol{v}_G}{\omega^2}. \text{ (True/False)}$$

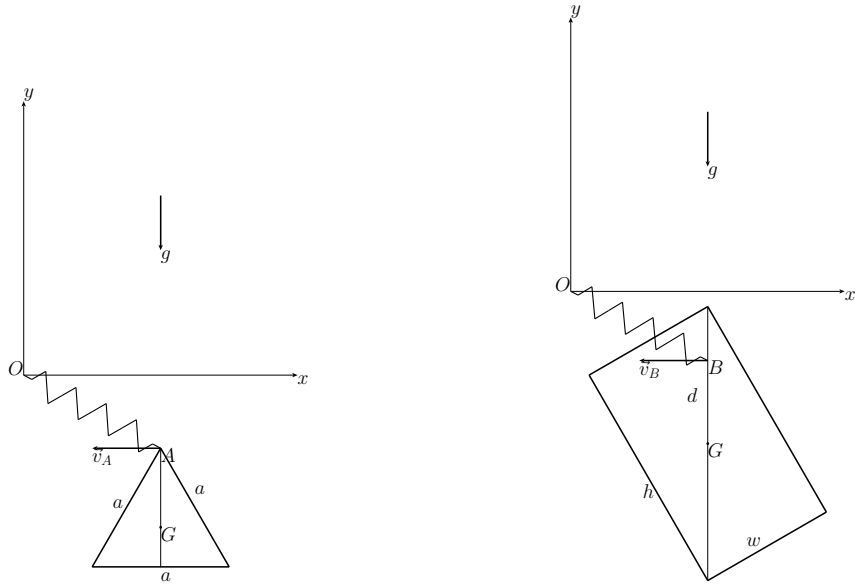
False. We are told that the body is in general motion. In general motion, $\boldsymbol{v}_G \cdot \boldsymbol{\omega} \neq \boldsymbol{0}$ and it is not possible to find a point with zero velocity. Only if the motion of the body is such that $\boldsymbol{v}_G \cdot \boldsymbol{\omega} = \boldsymbol{0}$, the statement would be true.

7. A heavy equilateral triangle of side a is in a vertical plane and its center of mass is constrained to move along the vertical line $x = 2a$, as shown in the figure. One of the vertices of the equilateral triangle, A, is attached to the origin by means of a spring with constant, k, and zero natural length. The position of the equilateral triangle is determined by the y coordinate of the center of mass, y_G^T, and the orientation on the vertical plane given by the angle θ^T, defined as the angle between GA and Oy. Initially, $y_G^T = -2a$, $\theta = 0, \dot{\theta}^T = \dot{\theta}_0$ and the velocity of the point A (see figure) is $\vec{v}_A(t=0) = -u\vec{i}$.

   On the other hand, a heavy rectangle of sides h and w is in a similar vertical plane and its center of mass constrained to move along the vertical line $x = 2a$, as shown in the figure. One point of the diagonal of the rectangle, B, is attached to the origin by means of a spring with constant, k, and zero natural length. The distance of this point to the center of mass is d. The position of the rectangle is determined by the y coordinate of the center of mass y_G^R and the orientation on the vertical plane given by the angle θ^R, defined as the angle between GB and Oy. Initially, the position of the rectangle is as it is shown in the figure, with $y_G^R = -2a$, $\theta^R = 0, \dot{\theta}^R = \dot{\theta}_0$ and the velocity of the point B, $\vec{v}_B(t=0) = -u\vec{i}$.

   - If the initial velocity of the center of mass G is the same for both triangle and rectangle, what is the value of d as a function of a?
   - If both rigid bodies are homogeneous with the same density, ρ, determine the conditions that should be satisfied by h and w as a function of a in order to have $y_G^T(t) \equiv y_G^R(t)$ and $\theta^T(t) \equiv \theta^R(t)$.

     Note 1: It is not necessary to solve the whole problem to answer the question.
     Note 2: Moment of inertia of an equilateral triangle of side a and mass m with respect to the center of mass: $I_G = ma^2/12$.
     Note 3: Moment of inertia of a rectangle of sides h, w and mass m w.r.t. the center of mass: $I_G = m(h^2 + w^2)/12$



The initial velocity of the center of mass can be computed using the velocity field of the rigid body and the velocity of A and B, respectively:

$$\boldsymbol{v}_A = -u\boldsymbol{i} = \dot{y}_G^T\boldsymbol{j} + \dot{\theta}_0\boldsymbol{k} \times |\boldsymbol{GA}|\boldsymbol{j} \tag{6}$$

$$\boldsymbol{v}_B = -u\boldsymbol{i} = \dot{y}_G^R\boldsymbol{j} + \dot{\theta}_0\boldsymbol{k} \times |\boldsymbol{GB}|\boldsymbol{j} \tag{7}$$

In order to have the same initial velocity of the center of mass, the condition $d = |\boldsymbol{GB}| = |\boldsymbol{GA}|$ must be satisfied.

The position of the center of mass of the equilateral triangle is at one third of the height of the triangle. The height of the triangle is $h = \sqrt{3}/2a$. Thus, $|\boldsymbol{GA}| = 1/\sqrt{3}a$ and $d = 1/\sqrt{3}a$.

The evolution of y_G and θ in both cases is governed by the linear momentum equation in the vertical direction and the angular momentum equation in the direction perpendicular to the plane of motion. That is:

$$m\ddot{y}_G = -m\,g - k\,(y_G + d\,\cos\theta) \tag{8}$$

$$I_z\ddot{\theta} = k\,d\,[y_G \sin\theta + 2\,a\cos\theta] \tag{9}$$

The parameters a and $d = 1/\sqrt{3}a$ are equal in both cases, and k is also the same. Therefore, in order to have the same evolution, both mass and moment of inertia must be equal.

$$\frac{\sqrt{3}}{4}a^2 = w\,h \tag{10}$$

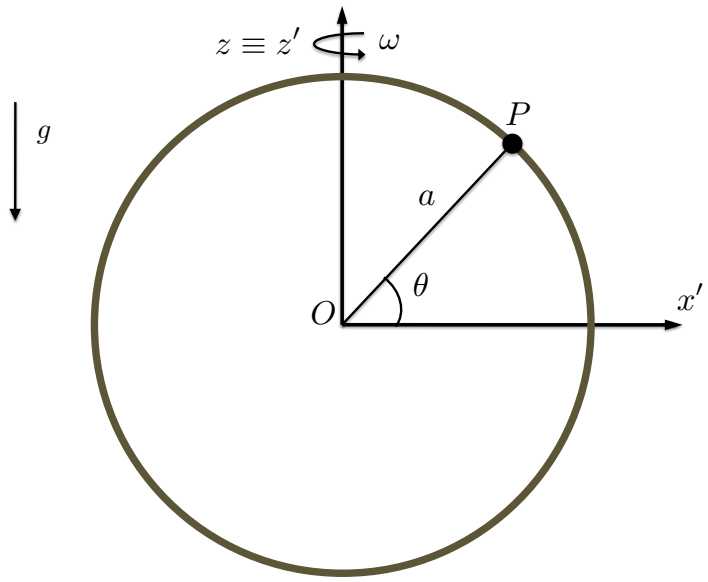
$$\frac{a^2}{12} = \frac{h^2 + w^2}{12} \tag{11}$$

This yields $h^2 + w^2 - 4/\sqrt{3}wh = 0$, and two possible solutions:

$$\text{a)} h = \frac{a}{2} \qquad w = \frac{\sqrt{3}}{2}a \tag{12}$$

$$\text{b)} h = \frac{\sqrt{3}}{2}a \qquad w = \frac{a}{2} \tag{13}$$

Problem 1



A vehicle P of mass m is constrained to move along a circular track of radius a contained in a vertical plane $Ox'z'$. We consider an inertial reference frame $Oxyz$(not shown in the figure). The vertical plane $Ox'z'$ containing the track rotates at a constant rate ω with respect to the axis $Oz \equiv Oz'$. The vehicle is subject to its own weight. In addition, the vehicle is subject to a repulsion force from the axis Oz, which is proportional to the distance to the axis, with a constant of proportionality $\lambda > 0$. We define the axis Oy' such that the moving reference frame $Ox'y'z'$ is right-handed.

1. Determine the fictitious forces acting on the vehicle P as a function of m, a, ω, θ and its time derivatives.
2. Write down a set of three ordinary differential equations that once solved (with given initial conditions) would yield the position of the vehicle (with respect to $Ox'y'z'$) and the reaction force that the track exerts on the vehicle, both as a function of time. (Note that you do NOT need to solve the equations)
3. For the particular case, $\lambda = mg/a$, $\omega = \sqrt{g/a}$, determine the equilibrium position of the vehicle with respect to the moving reference frame.
4. Finally, the vehicle has an engine that gives it the possibility of having an additional thrust force in the direction of the relative velocity. Assume that λ and ω have the same values as in the previous question. Initially the vehicle is located at $\boldsymbol{r}'_P = a\boldsymbol{i}'$ with velocity $\boldsymbol{v}'_P = v_0\boldsymbol{k}'$.
   (a) Determine the thrust as a function of time required for the vehicle to move at constant relative speed v_0. In that case, determine also the value of the reaction that the track exerts on the vehicle as a function of time.
   (b) If the thrust that the engine provides is always in the direction of the relative velocity, is it possible for the vehicle to travel over the whole track at constant relative speed v_0?

In order to solve the problem we will use both Cartesian coordinates and polar coordinates with unit vectors $\boldsymbol{u_r}$ and $\boldsymbol{u_\theta}$. The transformation is given by

$$\boldsymbol{i'} = \cos\theta\boldsymbol{u_r} - \sin\theta\boldsymbol{u_\theta}, \tag{14}$$

$$\boldsymbol{k'} = \sin\theta\boldsymbol{u_r} + \cos\theta\boldsymbol{u_\theta}. \tag{15}$$

The angular velocity is $\boldsymbol{\omega} = \omega\boldsymbol{k'}$. The relative position, velocity and acceleration of the particle are:

$$\boldsymbol{r'} = a\boldsymbol{u_r}, \tag{16}$$

$$\boldsymbol{v'} = a\dot{\theta}\boldsymbol{u_\theta}, \tag{17}$$

$$\boldsymbol{a'} = -a\dot{\theta}^2\boldsymbol{u_r} + a\ddot{\theta}\boldsymbol{u_\theta}. \tag{18}$$

1. Only two terms are non-zero, the Coriolis force and the centrifugal force. They are given by

$$\boldsymbol{F}_{cor} = -2m\boldsymbol{\omega} \times \boldsymbol{v}' = 2ma\omega\dot{\theta}\sin\theta\boldsymbol{j'} \tag{19}$$

$$\boldsymbol{F}_{centr} = -m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}') = ma\omega^2\cos\theta\boldsymbol{i'} \tag{20}$$

2. The external forces acting in the system are the weight $\boldsymbol{F}_w = -mg\boldsymbol{k'}$, the repulsion force $\boldsymbol{F}_{rep} = \lambda x'\boldsymbol{i'} = \lambda a\cos\theta\boldsymbol{i'}$ and the normal reaction that has two components $\boldsymbol{N} = N_1\boldsymbol{u_r} + N_2\boldsymbol{j'}$. The resulting vector equation is

$$\boldsymbol{F}_w + \boldsymbol{F}_{rep} + \boldsymbol{N} + \boldsymbol{F}_{cor} + \boldsymbol{F}_{centr} = m\boldsymbol{a}'. \tag{21}$$

Projecting this vector equation along $\boldsymbol{u_r}$, $\boldsymbol{u_\theta}$ and $\boldsymbol{j'}$ we obtain:

$$-mg\sin\theta + (\lambda + m\omega^2)a\cos^2\theta + N_1 = -ma\dot{\theta}^2, \tag{22}$$

$$-mg\cos\theta - (\lambda + m\omega^2)a\sin\theta\cos\theta = ma\ddot{\theta}, \tag{23}$$

$$N_2 + 2ma\omega\dot{\theta}\sin\theta = 0. \tag{24}$$

Solving eq. (23) with suitable initial condition we would obtain $\theta(t)$ and substituing this into eqs. (22) and (24) we would obtain $N_1(t)$ and $N_2(t)$.

3. In order to obtain the equilibrium positions we set the right hand side of eq. (23) to 0 and substitute the given values of λ and ω obtaining:

$$\cos\theta + 2\sin\theta\cos\theta = 0. \tag{25}$$

This equation has four solutions. Two when $\cos\theta = 0$, $\theta = \pm\pi/2$. Two when $\sin\theta = -1/2$, $\theta = -\pi/6$ and $\theta = -5\pi/6$. These four solutions are the four equilibrium positions.

4. Now we need to consider an additional force, the thrust $F_{thrust} = T\boldsymbol{u_\theta}$ but the acceleration of the particle is known. Since the relative speed is constant v_0, $\dot{\theta}$ is also constant equal to $\dot{\theta} = v_0/a$. The relative acceleration is $\boldsymbol{a}' = -v_0^2/a\boldsymbol{u_r}$ and the angle θ is obtained integrating and imposing the initial condition $\theta(t) = v_0 t/a$.

   (a) In order to obtain the thrust we re-write eq. 23 including the thrust and taking into account that the tangential acceleration is zero. We obtain

$$-mg\cos\theta - (\lambda + m\omega^2)a\sin\theta\cos\theta + T = 0. \tag{26}$$

Substituting λ, ω and $\theta(t)$ we obtain the thrust as a function of time

$$T = mg\left(\cos\left(\frac{v_0 t}{a}\right) + \sin\left(\frac{2v_0 t}{a}\right)\right). \tag{27}$$

The first component of the reaction is given by

$$-mg\sin\theta + 2mg\cos^2\theta + N_1 = -mv_0^2/a, \tag{28}$$

so that we obtain

$$N_1 = -mv_0^2/a + mg\left(\sin\left(\frac{v_0 t}{a}\right) + 2\cos^2\left(\frac{v_0 t}{a}\right)\right), \tag{29}$$

and for the second component

$$N_2 = -2m\omega v_0\sin\left(\frac{v_0 t}{a}\right). \tag{30}$$

   (b) We need to evaluate the value of the obtained function $T(\theta(t))$ between 0 and 2π. If it is always positive then the answer to the question would be yes, otherwise the answer would be no since at some point you would need to have thrust in the direction opposite to the relative velocity. We need to decide whether the following function

$$T = mg(\cos(\theta) + \sin(2\theta)), \tag{31}$$

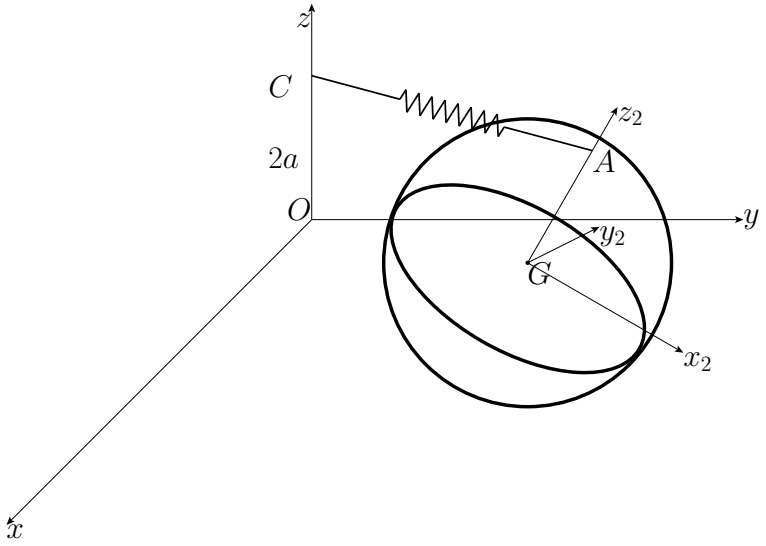
is positive between 0 and 2π. The answer is obviously no, since for example for $\theta = 3\pi/4$ we obtain $T < 0$.

Problem 2

The motion of a heavy sphere of mass M and radius a is to be analyzed. The sphere moves on a horizontal plane Oxy. In addition to the weight, a point A on its surface is attached to a point C on the vertical axis Oz by means of a spring with zero natural length and constant k. The distance $|OC|$ is equal to $2a$. An intermediate reference frame is defined in the following way: axis Gz_2 is the direction of the line connecting the center of mass of the sphere, G, and A; axis Gy_2 perpendicular to both Gz_2 and Oz and, therefore, parallel to the horizontal plane Oxy; and Gx_2 forming a right-handed reference frame. The position of the sphere is given by ξ, η, coordinates of the projection of G on the horizontal plane Oxy, angle ψ between Gy_2 and Oy, angle θ between Gz_2 and Oz, and angle φ corresponding to the rotation of the sphere about the Gz_2 axis. Note that the difference between the $Gx_2y_2z_2$ reference frame and a truly body-fixed reference frame is a rotation φ about the Gz_2 axis.

1. Considering the horizontal plane as a smooth surface:
   (a) express the angular momentum of the sphere with respect to G as a function of the variables of the problem,
   (b) express force and torque of the spring as a function of the variables of the problem,
   (c) obtain the set of five ordinary differential equations to solve the motion,
   (d) obtain also the reaction as a function of the above-mentioned variables.
2. If the horizontal plane is rough, and assuming that the sphere rolls without slipping, specify the changes on the set of required equations to solve the motion, the normal reaction and the friction force.

(The moment of inertia of a sphere of radius a and mass M with respect to the center of mass is $I_G = 3/5\, Ma^2$)



The acceleration of the center of mass in inertial coordinates is:

$$\boldsymbol{a}_G = \ddot{\xi}\boldsymbol{i} + \ddot{\eta}\boldsymbol{j} \tag{32}$$

In turn, the angular velocity of the sphere w.r.t. the inertial reference frame is:

$$\boldsymbol{\omega} = \dot{\psi}\boldsymbol{k} + \dot{\theta}\boldsymbol{j}_2 + \dot{\varphi}\boldsymbol{k}_2 \tag{33}$$

The vertical direction $\boldsymbol{k}$ is expressed in terms of its coordinates in the $Gx_2y_2z_2$ reference frame as:

$$\boldsymbol{k} = \cos\theta\boldsymbol{k}_2 - \sin\theta\boldsymbol{i}_2 \tag{34}$$

Therefore, the angular velocity of the sphere expressed in the $Gx_2y_2z_2$ reference frame has the form:

$$\boldsymbol{\omega} = -\dot{\psi}\sin\theta\boldsymbol{i}_2 + \dot{\theta}\boldsymbol{j}_2 + \left(\dot{\varphi} + \dot{\psi}\cos\theta\right)\boldsymbol{k}_2 \tag{35}$$

The moment of inertia of any of the three axis Gx_2, Gy_2 and Gz_2 ($I_{x_2}, I_{y_2}, I_{z_2}$) are equal. The sum of the three is equal to twice the moment of inertia with respect to G: $2\,I_G = I_{x_2} + I_{y_2} + I_{z_2}$. Therefore, the tensor of inertia in G is:

$$\bar{\bar{I}}_G = \frac{2}{5} M a^2 \bar{\bar{U}} \tag{36}$$

Thus, the angular momentum with respect to G is (in $Gx_2y_2z_2$ coordinates):

$$\boldsymbol{H}_G = \frac{2}{5} M a^2 \left[-\dot{\psi} \sin\theta \boldsymbol{i}_2 + \dot{\theta} \boldsymbol{j}_2 + \left(\dot{\varphi} + \dot{\psi} \cos\theta \right) \boldsymbol{k}_2 \right] \tag{37}$$

On the other hand, the forces that should be included are: weight and force and torque of the spring. Weight:

$$\boldsymbol{W} = -Mg\boldsymbol{k} \tag{38}$$

Force of the spring:

$$\boldsymbol{F}_s = -k\boldsymbol{CA} \tag{39}$$

where: $\boldsymbol{CA} = \boldsymbol{CO} + \boldsymbol{OG} + \boldsymbol{GA}$. Each vector is written as:

$$\begin{aligned} \boldsymbol{CO} &= -2\,a\boldsymbol{k} && (40)\\ \boldsymbol{OG} &= \xi\boldsymbol{i} + \eta\boldsymbol{j} + a\boldsymbol{k} && (41)\\ \boldsymbol{GA} &= a\boldsymbol{k}_2 && (42) \end{aligned}$$

Using: $\boldsymbol{k}_2 = \sin\theta\cos\psi\boldsymbol{i} + \sin\theta\sin\psi\boldsymbol{j} + \cos\theta\boldsymbol{k}$, the force of the spring in inertial coordinates is:

$$\boldsymbol{F}_m = -k\,a \left[\left(\frac{\xi}{a} + \sin\theta\cos\psi \right) \boldsymbol{i} + \left(\frac{\eta}{a} + \sin\theta\sin\psi \right) \boldsymbol{j} + (\cos\theta - 1)\,\boldsymbol{k} \right] \tag{43}$$

Moreover, the torque is computed as:

$$\boldsymbol{M}_m = \boldsymbol{GA} \times \boldsymbol{F}_m = \boldsymbol{GA} \times -k\,(\boldsymbol{CG} + \boldsymbol{GA}) = -k\,\boldsymbol{GA} \times \boldsymbol{CG} \tag{44}$$

Taking into account that $\boldsymbol{CG} = \xi\boldsymbol{i} + \eta\boldsymbol{j} - a\boldsymbol{k}$ and using the transformation expressions:

$$\begin{aligned} \boldsymbol{i} &= \cos\psi\cos\theta\boldsymbol{i}_2 - \sin\psi\boldsymbol{j}_2 + \cos\psi\sin\theta\boldsymbol{k}_2 && (45)\\ \boldsymbol{j} &= \sin\psi\cos\theta\boldsymbol{i}_2 + \cos\psi\boldsymbol{j}_2 + \sin\psi\sin\theta\boldsymbol{k}_2 && (46)\\ \boldsymbol{k} &= -\sin\theta\boldsymbol{i}_2 + \cos\theta\boldsymbol{k}_2 && (47) \end{aligned}$$

the torque of the spring in coordinates $Gx_2y_2z_2$ is:

$$\boldsymbol{M}_m = k\,a^2 \left[\left(\frac{\eta}{a}\cos\psi - \frac{\xi}{a}\sin\psi \right) \boldsymbol{i}_2 - \left(\frac{\xi}{a}\cos\psi\cos\theta + \frac{\eta}{a}\sin\psi\cos\theta + \sin\theta \right) \boldsymbol{j}_2 \right] \tag{48}$$

In turn, the time derivative of the angular momentum is computed as:

$$\left.\frac{\mathrm{d}\boldsymbol{H}_G}{\mathrm{d}t}\right|_2 + \boldsymbol{\omega}_2 \times \boldsymbol{H}_G = \frac{2}{5} M a^2 \left[\left(-\ddot{\psi}\sin\theta + \dot{\psi}\dot{\theta}\cos\theta + \dot{\theta}\dot{\varphi} \right) \boldsymbol{i}_2 + \left(\ddot{\theta} + \dot{\varphi}\dot{\psi}\sin\theta \right) \boldsymbol{j}_2 + \left(\dot{\varphi} + \dot{\psi}\cos\theta \right) \boldsymbol{k}_2 \right] \tag{49}$$

where $\boldsymbol{\omega}_2 = -\dot{\psi}\sin\theta\boldsymbol{i}_2 + \dot{\theta}\boldsymbol{j}_2 + \dot{\psi}\cos\theta\boldsymbol{k}_2$ is the angular velocity of the reference frame $Gx_2y_2z_2$.

Therefore, the set of equations to solve for the motion is:

$$\begin{aligned} M\ddot{\xi} &= -k\,(\xi + a\sin\theta\cos\psi) && (50)\\ M\ddot{\eta} &= -k\,(\eta + a\sin\theta\sin\psi) && (51)\\ \frac{2}{5}Ma^2\left(-\ddot{\psi}\sin\theta + \dot{\psi}\dot{\theta}\cos\theta + \dot{\theta}\dot{\varphi} \right) &= k\,a\,(\eta\cos\psi - \xi\sin\psi) && (52)\\ \frac{2}{5}Ma^2\left(\ddot{\theta} + \dot{\varphi}\dot{\psi}\sin\theta \right) &= -k\,a^2\left(\frac{\xi}{a}\cos\psi\cos\theta + \frac{\eta}{a}\sin\psi\cos\theta + \sin\theta \right) && (53)\\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{\varphi} + \dot{\psi}\cos\theta \right) &= 0 && (54) \end{aligned}$$

Concerning the computation of the reaction, the equation of the linear momentum in the vertical direction yields:

$$N = M\,g - ka(1 - \cos\theta) \tag{55}$$

When the horizontal plane is rough, friction must be considered. Additionally, we have a constraint due to the condition of rolling without slipping. If we call I the point of contact between sphere and plane, the constraint reads: $\boldsymbol{v}_I = \boldsymbol{0}$. The point I belongs to the sphere and, therefore, the velocity field of the sphere can be used:

$$\boldsymbol{v}_I = \boldsymbol{v}_G + \boldsymbol{\omega} \times \boldsymbol{GI} = \boldsymbol{0} \tag{56}$$

$\boldsymbol{GI} = -a\boldsymbol{k}$ and $\boldsymbol{v}_G = \dot{\xi}\boldsymbol{i} + \dot{\eta}\boldsymbol{j}$. The angular velocity in inertial coordinates is written as:

$$\boldsymbol{\omega} = \left(\dot{\varphi}\sin\theta\cos\psi - \dot{\theta}\sin\psi\right)\boldsymbol{i} + \left(\dot{\theta}\cos\psi + \dot{\varphi}\sin\theta\sin\psi\right)\boldsymbol{j} + \left(\dot{\psi} + \dot{\varphi}\cos\theta\right)\boldsymbol{k} \tag{57}$$

Therefore, the rolling condition as a function of the variables of the problem reads:

$$\dot{\xi} - a\left(\dot{\varphi}\sin\theta\sin\psi + \dot{\theta}\cos\psi\right) = 0 \tag{58}$$

$$\dot{\eta} - a\left(\dot{\theta}\sin\psi - \dot{\varphi}\sin\theta\cos\psi\right) = 0 \tag{59}$$

On the other hand, eq (50-51) must be modified to include friction:

$$M\ddot{\xi} = -k\left(\xi + a\sin\theta\cos\psi\right) + R_x \tag{60}$$

$$M\ddot{\eta} = -k\left(\eta + a\sin\theta\sin\psi\right) + R_y \tag{61}$$

where R_x and R_y are the components of the friction in Ox and Oy, respectively. Finally, the torque due to friction also enters the equation of the angular momentum. The torque expressed in both systems of coordinates is:

$$\boldsymbol{M}_f = a\left(-R_y\boldsymbol{i} + R_x\boldsymbol{j}\right) = a\left[\left(R_x\sin\psi - R_y\cos\psi\right)\left(\cos\theta\boldsymbol{i}_2 + \sin\theta\boldsymbol{k}_2\right) + \left(R_x\cos\psi + R_y\sin\psi\right)\boldsymbol{j}_2\right] \tag{62}$$

and the equation of the angular momentum is rewritten as:

$$\begin{aligned}
\frac{2}{5}\left(\ddot{\psi}\sin\theta + \dot{\psi}\dot{\theta}\cos\theta + \dot{\theta}\dot{\varphi}\right) &= \frac{k}{M}\left(\frac{\eta}{a}\cos\psi - \frac{\xi}{a}\sin\psi\right) + \left(\frac{R_x}{Ma}\sin\psi - \frac{R_y}{Ma}\cos\psi\right)\cos\theta \\
\frac{2}{5}\left(\ddot{\theta} - \dot{\varphi}\dot{\psi}\sin\theta\right) &= \frac{k}{M}\left(\frac{\xi}{a}\cos\psi\cos\theta + \frac{\eta}{a}\sin\psi\cos\theta - \sin\theta\right) + \left(\frac{R_x}{Ma}\cos\psi + \frac{R_y}{Ma}\sin\psi\right) \\
\frac{2}{5}\left(\ddot{\varphi} + \ddot{\psi}\cos\theta - \dot{\psi}\dot{\theta}\sin\theta\right) &= \left(\frac{R_x}{Ma}\sin\psi - \frac{R_y}{Ma}\cos\psi\right)\sin\theta
\end{aligned}$$

Note: This problem has been solved in the previous lines using the intermediate reference frame $Gx_2y_2z_2$ to project the components of the angular momentum. However, the problem could have been solved in a similar manner using a body-fixed reference frame to project $\boldsymbol{H}_G$. In this problem, even the inertial reference frame would be useful for this purpose, given that the time derivative of the inertia tensor of the sphere is zero in the inertial reference frame.