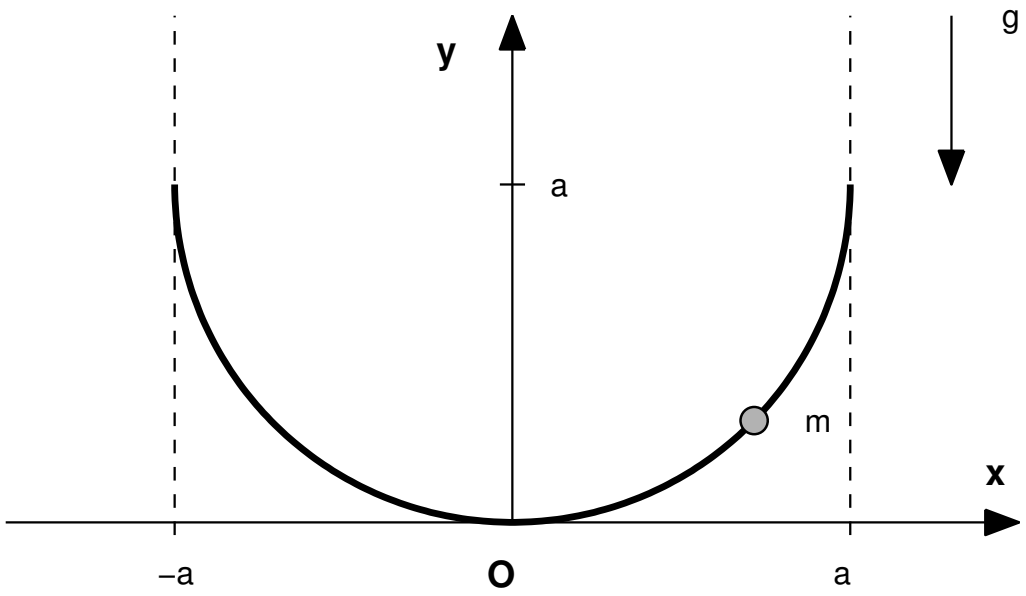
1. A heavy point particle P of mass m is constrained to move without friction along the half-circumference of radius a shown in the figure. Initially, the particle is located at $x = 0$ with velocity v_0 along the positive x axis.



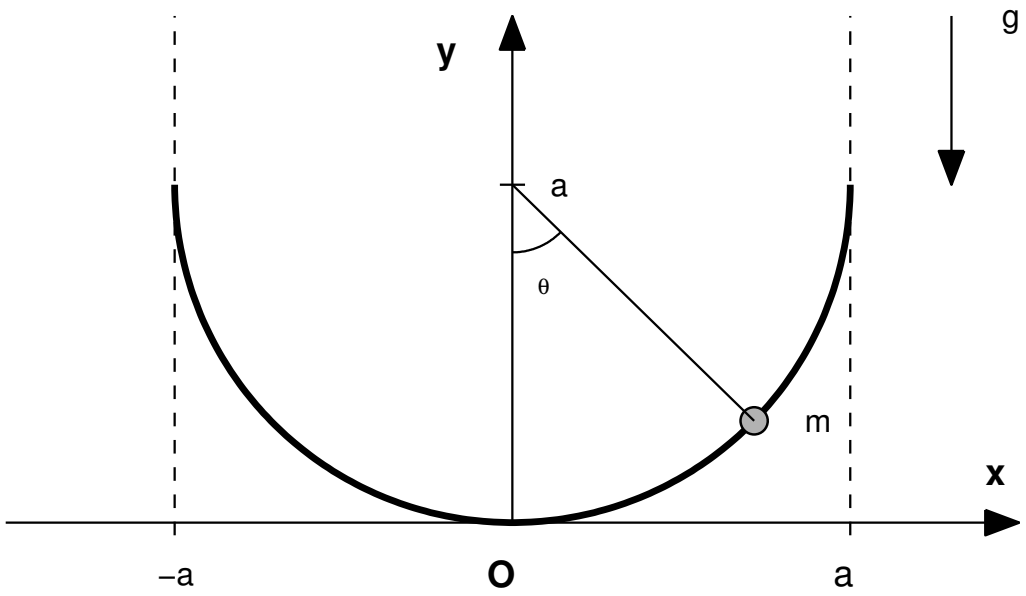
- Determine the number of degrees of freedom of the problem.
- Reduce the problem to quadratures.
- What should be the minimum value of v_0 in order to reach $x = a$? How much time would it take to reach that point?

Let's assume now that the curve containing the point particle rotates around the OX axis with constant angular velocity $\boldsymbol{\omega} = \Omega \boldsymbol{i}$.

- Define a convenient moving reference frame with origin in O, $OX_1Y_1Z_1$.
- Express the relative velocity and relative acceleration with respect to $OX_1Y_1Z_1$.
- Determine the inertia force that acts on the particle with respect to $OX_1Y_1Z_1$.
- Express also the components of the weight of the particle with respect to $OX_1Y_1Z_1$.
- Obtain a system of three coupled ordinary differential equations whose solution would provide the position of the particle and the value of the reaction force as a function of time. (Note: the solution of the system is not being asked for)

- The point particle P moves within a plane. This corresponds to two degrees of freedom. The particle is constrained to move along a curve. This constraint removes one of the degrees of freedom but adds a reaction force $\boldsymbol{R}$ along the normal direction to the curve. The position of the particle is completely specified by the coordinate x. In the present case, as the curve is a half-circumference, it is easier to use the parameter θ as shown in the figure. The parameter θ specifies also completely the position of the particle, since

$$x = a\sin\theta \tag{1}$$
$$y = a(1-\cos\theta) \tag{2}$$



The particle is subject to two forces: its own weight and the reaction. The reaction does not perform work and the weight derives from a potential

$$V = mgy = mga\,(1-\cos\theta)\,, \tag{3}$$

therefore, the energy is conserved. Since there is only one degree of freedom and the energy is conserved, it is possible to use the equation of conservation of energy, $T + V = E$, to solve the problem. The kinetic energy can be expressed as

$$T = \frac{1}{2}m\left(\dot{x}^2+\dot{y}^2\right) = \frac{1}{2}ma^2\dot{\theta}^2, \tag{4}$$

and the total energy E is obtained imposing the initial condition $E = (1/2)mv_0^2$. Thus,

$$\dot{\theta}^2 + \frac{2g}{a}\,(1-\cos\theta) = \frac{v_0^2}{a^2}. \tag{5}$$

Defining $G_1(\theta)$ as

$$G_1(\theta) = \frac{v_0^2}{a^2} - \frac{2g}{a}\,(1-\cos\theta)\,, \tag{6}$$

equation (5) leads to:

$$\dot{\theta}^2 = G_1(\theta) \rightarrow t = \int_0^{\theta} \frac{\pm \mathrm{d}\theta}{\sqrt{G_1(\theta)}}. \tag{7}$$

- The maximum and minimum values of x are obtained when $\dot{x}$ is 0. This condition is equivalent to $\dot{\theta} = 0$. From equation (5) this happens when

$$\cos\theta = 1 - \frac{v_0^2}{2ga}. \tag{8}$$

If $x = a$, then $\theta = \pi/2$. Imposing the latter in equation (8), the result is $v_0 = \sqrt{2ga}$. If $v_0 = \sqrt{2ga}$, $G_1(\pi/2) = 0$ and from equation (7), it is possible to infer that the particle will reach $x = a$ when $t \rightarrow \infty$.

- We define the moving reference frame such that 1) the axis OX_1 coincides at all times with OX 2) the axis OY_1 is contained in the plane OYZ and rotates at a constant angular velocity $\boldsymbol{\omega} = \Omega \boldsymbol{i}$ and 3) the axis OZ_1 is orthogonal to OX_1 and OY_1 forming a righ-handed coordinate system.
- In this coordinate system the relative position of P is given by

$$\boldsymbol{r_1} = x_1 \boldsymbol{i_1} + y_1 \boldsymbol{j_1} = a \sin\theta \boldsymbol{i_1} + a\,(1 - \cos\theta)\,\boldsymbol{j_1} \tag{9}$$

or introducing polar coordinates on the plane OX_1Y_1 with origin in $A = (0, a)$

$$\boldsymbol{r_1} = a\boldsymbol{j_1} + a\boldsymbol{u_r}. \tag{10}$$

The relative velocity is then

$$\boldsymbol{v_1} = a\dot{\theta}\boldsymbol{u_\theta} \tag{11}$$

and the relative acceleration

$$\boldsymbol{a_1} = a\ddot{\theta}\boldsymbol{u_\theta} - a\dot{\theta}^2\boldsymbol{u_r}. \tag{12}$$

In the following it will be useful the expression of $\boldsymbol{j_1}$ as a function of $\boldsymbol{u_r}$ and $\boldsymbol{u_\theta}$,

$$\boldsymbol{j_1} = -\cos\theta \boldsymbol{u_r} + \sin\theta \boldsymbol{u_\theta}. \tag{13}$$

- Only two of the inertia force terms are non-zero: the centrifugal force and the Coriolis force. The centrifugal force can be expressed

$$\boldsymbol{F_{centr}} = -m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r_1}) = m\Omega^2 a(1 - \cos\theta)\boldsymbol{j_1}, \tag{14}$$

and the Coriolis force

$$\boldsymbol{F_{cor}} = -2m\boldsymbol{\omega} \times \boldsymbol{v_1} = -2m\Omega a\dot{\theta}\sin\theta \boldsymbol{k_1}. \tag{15}$$

- The weight of the particle is

$$\boldsymbol{W} = -mg\boldsymbol{j} = -mg(\cos(\Omega t)\boldsymbol{j_1} - \sin(\Omega t)\boldsymbol{k_1}). \tag{16}$$

- The reaction force has two components

$$\boldsymbol{R} = R_1\boldsymbol{u_r} + R_2\boldsymbol{k_1}, \tag{17}$$

since the normal to the curve is always parallel to $\boldsymbol{u_r}$

In order to obtain the three equations, we need to use relative dynamics.

$$\sum \boldsymbol{F} + \sum \boldsymbol{F_I} = m\boldsymbol{a_1}, \tag{18}$$

where $\sum \boldsymbol{F}$ is the sum of the external forces (weight and reaction) and $\sum \boldsymbol{F_I}$ is the sum of the inertia forces.

The equation for θ is obtained by projecting along $\boldsymbol{u_\theta}$

$$ma\ddot{\theta} = -mg\sin\theta\cos(\Omega t) + m\Omega^2 a\sin\theta(1-\cos\theta) \tag{19}$$

that can be re-written:

$$\ddot{\theta} = \Omega^2\sin\theta\left(1 - \frac{g}{a\Omega^2}\cos(\Omega t) - \cos\theta\right). \tag{20}$$

Integrating twice (using a numerical method for example) we would obtain $\theta(t)$ and therefore the position of the particle would be specified as a function of time.

Once $\theta(t)$ is known, proyecting along $\boldsymbol{u_r}$ we obtain $R_1(t)$

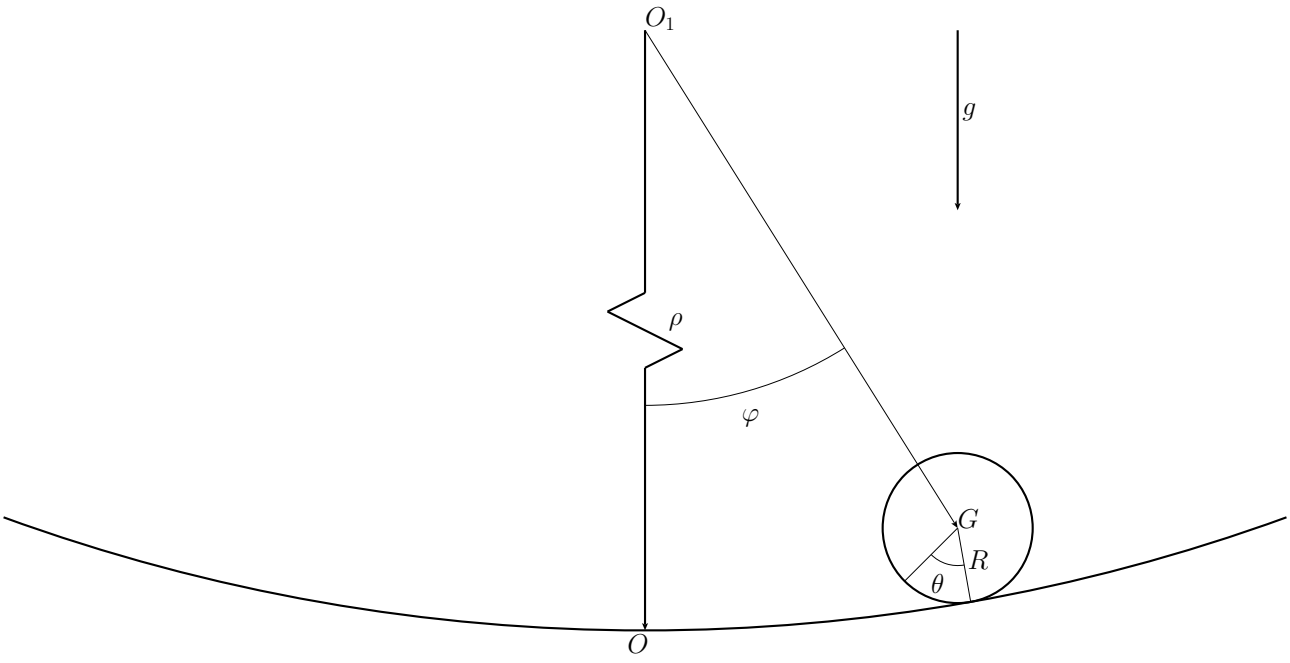
$$R_1 = ma\Omega^2\cos\theta(1-\cos\theta) - mg\cos(\Omega t)\cos\theta - ma\dot{\theta}^2 \tag{21}$$

and proyecting along $\boldsymbol{k_1}$ we obtain $R_2(t)$

$$R_2 = 2ma\Omega\dot{\theta}\sin\theta - mg\sin(\Omega t). \tag{22}$$

2. A disc of radius R rolls without slipping on a circumference of radius ρ, as shown in the figure. The mass of the disc is m and it is subject to the action of gravity. Initially, the velocity of the center of mass is zero and the height of the center of mass (with respect the lowest point of the surface) is $2R$.

   - Reduce the motion to quadratures
   - Indicate an equilibrium position of the previous system
   - Assuming the oscillations around the equilibrium position are small and the disc is left initially at rest near the equilibrium position, solve the motion. (In the approximation, it is considered that: $\varphi \ll 1$, $\cos(\varphi) \approx 1 - \varphi^2/2$, $\sin(\varphi) \approx \varphi$)



(a) The condition of being on the cylindrical surface is imposed in a similar way as in the case of a planar surface. The position of the center of mass of the system can be expressed as a function of only one variable, but in this case we choose, intead of the x position, the angle φ which describes the position of the center of mass of the disc in the circumference of the cylinder. Thus,

$$\boldsymbol{OG} = (\rho - R)\sin\varphi\boldsymbol{i} + [\rho(1-\cos\varphi) + R\cos\varphi]\,\boldsymbol{j} \tag{1}$$

Differenciating the previous expression in the inertial reference frame, we obtain the velocity of the center of mass as a function of φ and $\dot{\varphi}$:

$$\boldsymbol{v}_G = (\rho - R)\dot{\varphi}\,(\cos\varphi\boldsymbol{i} + \sin\varphi\boldsymbol{j}) \tag{2}$$

Using the condition of rolling without slipping, we are able to relate the position of the center of mass (φ) to the turned angle θ. For doing that, we should take into account that the angular velocity is: $\boldsymbol{\omega} = -\dot{\theta}\boldsymbol{k}$. The condition reads:

$$\boldsymbol{v}_I = \boldsymbol{v}_G + \boldsymbol{\omega} \times \boldsymbol{GI} = \boldsymbol{0} \tag{3}$$

The value of the vector $\boldsymbol{GI}$ can be obtained as the differece: $\boldsymbol{GI} = \boldsymbol{OI} - \boldsymbol{OG} = R(\sin\varphi\boldsymbol{i} - \cos\varphi\boldsymbol{j})$, where $\boldsymbol{OI} = \rho\,(\sin\varphi\boldsymbol{i} + (1-\cos\varphi)\boldsymbol{j})$. Thus, the condition of rolling without slipping leads to the following relationship:

$$\dot{\theta} = \left(\frac{\rho}{R} - 1\right)\dot{\varphi} \tag{4}$$

At this point, the Newton-Euler equations of motion can be posed, obtaining a system with three ODEs of second order and three unknowns: two reactions N and f and θ. Nevertheless, the best way to go is to pose the energy conservation. This can be used because the external force derives from a potential and the reactions do not perform work.

$$\frac{1}{2}mv_G^2 + \frac{1}{2}I_G\dot{\theta}^2 + mgy_G = E_0 \tag{5}$$

$$\frac{1}{2}m(\rho - R)^2\dot{\varphi}^2 + \frac{1}{2}I_G\dot{\theta}^2 + mg\,[\rho(1-\cos\varphi) + R\cos\varphi] = E_0 \tag{6}$$

Initially $\dot{\theta} = 0$ and $y_G = 2R$. Thus, $E_0 = 2mgR$. In addition, substituting the value of $\dot{\theta}$:

$$(mR^2 + I_G)\left(\frac{\rho}{R} - 1\right)^2\dot{\varphi}^2 = 2mg\,(\rho(\cos\varphi - 1) + R(2 - \cos\varphi)) \tag{7}$$

This equation is a first order ODE in which the unknown are φ and $\dot{\varphi}$. Therefore, the problem is already reduced to quadratures, from (7):

$$\int \frac{\pm d\varphi}{\sqrt{(\rho(\cos\varphi - 1) + R(2 - \cos\varphi))}} = \frac{R}{\rho - R}\sqrt{\frac{2mg}{(mR^2 + I_G)}}t \tag{8}$$

(b) An equilibrium position requires that $\dot{\varphi}$ (and therefore $\dot{\theta}$) was equal to zero. This takes place for $y_G = R$ ($\varphi = 0$). The lowest point of the cylinder is an equilibrium position.

(c) Small oscillations around the equilibrium position means $\varphi \ll 1$ and $\cos(\varphi) \approx 1 - \varphi^2/2$, as it is said in the statement of the problem. Therefore,

$$y_G = \rho(1-\cos\varphi) + R\cos\varphi \approx R + (\rho - R)\varphi^2/2 \tag{9}$$

The initial value of the energy is: $E_0 = mgy_G(t=0) \approx R - (\rho - R)\varphi_0^2/2$. Using again the equation of the energy, we obtain:

$$(mR^2 + I_G)\left(\frac{\rho}{R} - 1\right)^2\dot{\varphi}^2 = mg(\rho - R)\left(\varphi_0^2 - \varphi^2\right) \tag{10}$$

And, finally we get an explicit solution of φ as a function of time:

$$\frac{\pm d\varphi}{\sqrt{\varphi_0^2 - \varphi^2}} = \sqrt{\frac{mgR^2}{(mR^2 + I_G)(\rho - R)}} dt \Rightarrow \varphi = \pm\varphi_0 \sin\left(\sqrt{\frac{mgR^2}{(mR^2 + I_G)(\rho - R)}} t\right) \tag{11}$$

3. We consider the planar motion in a vertical plane OXY of a heavy homogeneous rectangle of mass m, length a and width $b = \sqrt{3}a$. The motion of the rectangle is influenced by the spring of constant k which connects the origin of the inertial reference frame $O_1X_1Y_1$ to the vertex A of the rectangle, as shown in the figure. In addition, the vertex C of the rectangle is constrained to move along the horizontal line $y = 5a/4$.

   (a) Determine the moment of inertia with respect to the center of mass of the rectangle G.

   (b) If at time $t = 0$, the side BC of the rectangle is located on the axis O_1Y_1, the vertex A has a positive x_1 component and the velocity of the vertex A is $\boldsymbol{v}_A = u_0\boldsymbol{i}_1 + v_0\boldsymbol{j}_1$ (where $\boldsymbol{i}_1$,$\boldsymbol{j}_1$ are the unit vectors along the axes O_1X_1 and O_1Y_1).

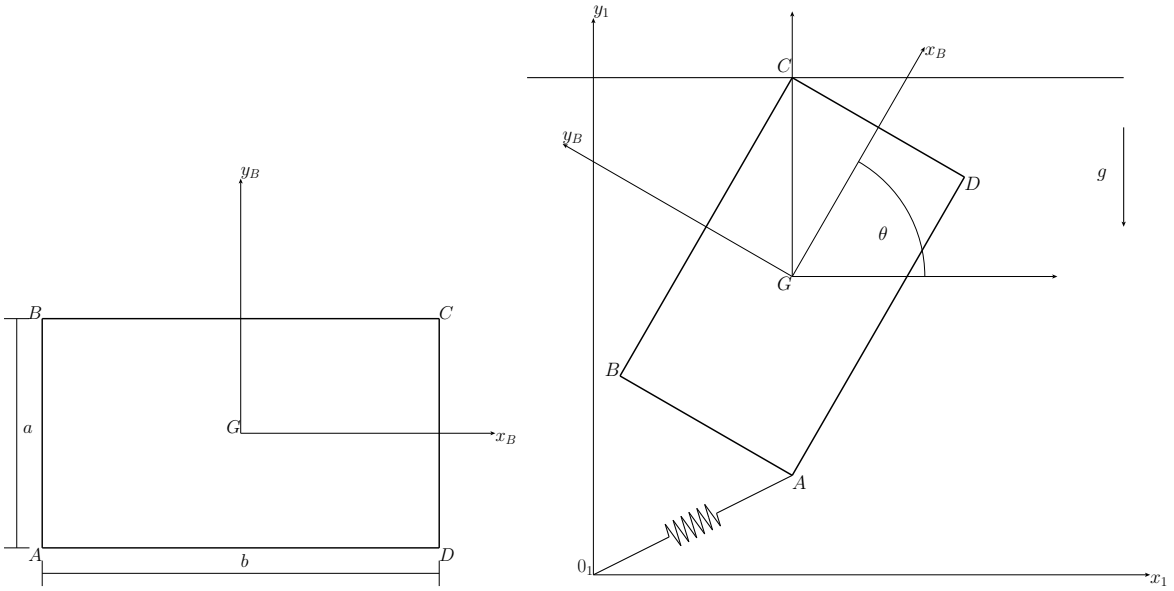
   - Write down the equations of motion for the rectangle.
   - Specify the initial conditions that need to be imposed on the previous equations in order to completely specify the problem. Determine also the instantaneous axis of rotation at $t = 0$.

   (c) If at time $t = 0$, the rectangle is at rest, the value of the angle is $\theta = \pi/3$ and we know that the body is in equilibrium. Determine the position of the centre of mass and the reaction force.

   (d) In the equilibrium position computed before, we now do the following actions:

   i. fix the position of G
   ii. release the vertex C from the constraint
   iii. apply a small perturbation to the orientation of the rectangle. Such that $\theta = \pi/3 + \phi$ and $\phi \ll 1$.

   Determine the frequency of oscillation of the rectangle with respect to the equilibrium position.



(a) The center of mass of the system, G, is at the geometrical center of the rectangle. The moment of inertia of the rectangle with respect to the center of mass can be computed in two steps. First, the moment of inertia with respect to the vertex A is computed, and second, the parallel axis theorem is used to obtain I_G. The moment of inertia with respect to the vertex A, I_A, is:

$$I_A = \int\int (x^2+y^2)dm = \frac{m}{ab}\int_0^a \int_0^b (x^2+y^2)dxdy = m\left[\frac{b^2}{3}+\frac{a^2}{3}\right] \tag{1}$$

Using the parallel axis theorem, we obtain:

$$I_G = I_A - m\left(\left(\frac{b}{2}\right)^2 + \left(\frac{a}{2}\right)^2\right) = \frac{m}{12}\left(b^2+a^2\right) = \frac{ma^2}{3} \tag{2}$$

(b) The equations of motion consist of two equations for the linear momentum of the rectangle and the equation of the angular momentum along the direction perpendicular to the plane of motion. The forces that are involved are: 1) weight; 2) force due to the spring; and 3) reaction. The latter two will produce a torque with respect to the center of mass, and therefore, they must be taken into account in the angular momentum equation. The variables that define the position of the rectangle are: x_G, y_G and θ. In a first step, the position of A and C are computed as a function of these variables.

$$\boldsymbol{O_1A} = \boldsymbol{O_1G} + \boldsymbol{GA} = x_G\boldsymbol{i}_1 + y_G\boldsymbol{j}_1 + \boldsymbol{GA} \tag{3}$$
$$\boldsymbol{O_1C} = \boldsymbol{O_1G} + \boldsymbol{GC} = x_G\boldsymbol{i}_1 + y_G\boldsymbol{j}_1 + \boldsymbol{GC} \tag{4}$$

And

$$\boldsymbol{GA} = -\frac{b}{2}\boldsymbol{i}_B - \frac{a}{2}\boldsymbol{j}_B = \left(-\frac{b}{2}\cos\theta + \frac{a}{2}\sin\theta\right)\boldsymbol{i}_1 - \left(\frac{b}{2}\sin\theta + \frac{a}{2}\cos\theta\right)\boldsymbol{j}_1 \tag{5}$$
$$\boldsymbol{GC} = \frac{b}{2}\boldsymbol{i}_B + \frac{a}{2}\boldsymbol{j}_B = \left(\frac{b}{2}\cos\theta - \frac{a}{2}\sin\theta\right)\boldsymbol{i}_1 + \left(\frac{b}{2}\cos\theta + \frac{a}{2}\sin\theta\right)\boldsymbol{j}_1 \tag{6}$$

where the relations: $\boldsymbol{i}_B = \cos\theta\boldsymbol{i}_1 + \sin\theta\boldsymbol{j}_1$, $\boldsymbol{j}_B = -\sin\theta\boldsymbol{i}_1 + \cos\theta\boldsymbol{j}_1$ have been used.
Therefore, the constrain on the vertex C can be expressed as a function of x_G, y_G and θ as:

$$\frac{5a}{4} = y_G + \frac{b}{2}\sin\theta + \frac{a}{2}\cos\theta \tag{7}$$

The corresponding reaction force is directed along O_1Y_1.
The force and torque due to the spring are computed as:

$$\boldsymbol{F}_S = -k\boldsymbol{x}_A = -k\left[\left(x_G - \frac{b}{2}\cos\theta + \frac{a}{2}\sin\theta\right)\boldsymbol{i}_1 + \left(y_G - \frac{b}{2}\sin\theta - \frac{a}{2}\cos\theta\right)\boldsymbol{j}_1\right] \tag{8}$$

$$\boldsymbol{T}_S = (\boldsymbol{x}_A - \boldsymbol{x}_G)\times\boldsymbol{F}_S = \boldsymbol{x}_G \times k\boldsymbol{x}_A = k\begin{vmatrix}\boldsymbol{i}_1 & \boldsymbol{j}_1 & \boldsymbol{k}_1\\ x_G & y_G & 0\\ x_A & y_A & 0\end{vmatrix} = k(x_Gy_A - y_Gx_A)\boldsymbol{k} \tag{9}$$

In turn, the force and torque due to the reaction force are computed as:

$$\boldsymbol{F}_R = R\boldsymbol{j}_1 \tag{10}$$

$$\boldsymbol{T}_R = \boldsymbol{GC} \times \boldsymbol{F}_R = \begin{vmatrix} \boldsymbol{i}_1 & \boldsymbol{j}_1 & \boldsymbol{k}_1 \\ \frac{b}{2}\cos\theta - \frac{a}{2}\sin\theta & \frac{b}{2}\cos\theta + \frac{a}{2}\sin\theta & 0 \\ 0 & R & 0 \end{vmatrix} = R\left(\frac{b}{2}\cos\theta - \frac{a}{2}\sin\theta\right)\boldsymbol{k}_1 \quad (11)$$

Finally, the equations of motion take the form:

$$\begin{aligned} m\ddot{x}_G &= -k\left(x_G - \frac{b}{2}\cos\theta + \frac{a}{2}\sin\theta\right) \\ m\ddot{y}_G &= -k\left(y_G - \frac{b}{2}\cos\theta - \frac{a}{2}\sin\theta\right) - mg + R \\ I_G\ddot{\theta} &= \frac{k}{2}\left[b\left(-x_G\sin\theta + y_G\cos\theta\right) - a\left(x_G\cos\theta + y_G\sin\theta\right)\right] + R\left(\frac{b}{2}\cos\theta - \frac{a}{2}\sin\theta\right) \\ y_G + \frac{b}{2}\sin\theta + \frac{a}{2}\cos\theta &= \frac{5a}{4} \end{aligned}$$

Regarding the required initial conditions, we know that at $t = 0$:

$$\begin{aligned} x_G(t=0) &= \frac{a}{2} \\ y_G(t=0) &= \frac{5}{4}a - \frac{b}{2} \\ \theta(t=0) &= \frac{\pi}{2} \end{aligned}$$

and $\boldsymbol{v}_A$. Using the velocity field of the rigid body, the initial conditions for the time derivatives of the state variables are obtained:

$$\boldsymbol{v}_A(t=0) = \boldsymbol{v}_G + \boldsymbol{\omega}\times\boldsymbol{GA} = \dot{x}_G\boldsymbol{i}_1 + \dot{y}_G\boldsymbol{j}_1 + \dot{\theta}\left[\left(-\frac{b}{2}\cos\theta + \frac{a}{2}\sin\theta\right)\boldsymbol{j}_1 + \left(\frac{b}{2}\sin\theta + \frac{a}{2}\cos\theta\right)\boldsymbol{i}_1\right] \quad (12)$$

And therefore:

$$\begin{aligned} u_0 &= \dot{x}_G(t=0) + \frac{b}{2}\dot{\theta}(t=0) \\ v_0 &= \dot{y}_G(t=0) + \frac{a}{2}\dot{\theta}(t=0) \end{aligned}$$

The last initial condition is obtained by differentiation of the constraint:

$$0 = \dot{y}_G + \frac{b}{2}\cos\theta\dot{\theta} - \frac{a}{2}\sin\theta\dot{\theta} \Rightarrow \dot{y}_G(t=0) = \frac{a}{2}\dot{\theta}(t=0)$$

Note that $y_G(t=0)$ and $\theta(t=0)$ fulfill the constraint $y_C = 5a/4$. Finally, we get:

$$\begin{aligned} \dot{x}_G(t=0) &= u_0 - \frac{\sqrt{3}}{2}v_0 \\ \dot{y}_G(t=0) &= \frac{v_0}{2} \\ \dot{\theta}(t=0) &= \frac{v_0}{a} \end{aligned}$$

In planar motion, the instantaneous axis of rotation is perpendicular to the plane of motion and the point P in which the axis intersects the plane has zero velocity. Therefore, using the velocity field of the rigid body, we can compute the position of P:

$$\boldsymbol{v}_P = \boldsymbol{v}_A + \boldsymbol{\omega}\times\boldsymbol{AP} = \boldsymbol{0}$$

Then, $\boldsymbol{AP}\cdot\boldsymbol{i} = -v_0/\dot{\theta}$, $\boldsymbol{AP}\cdot\boldsymbol{j} = u_0/\dot{\theta}$ and finally, $\boldsymbol{O_1P} = \boldsymbol{O_1A} + \boldsymbol{AP}$.

(c) At the equilibrium position, the equations of motion must be fulfilled with all the time derivatives equal to zero. Moreover, the constraint on the vertex C must also be satisfied. Thus, taken into account that $\theta^{(E)} = \pi/3$:

$$\frac{5a}{4} = y_G^{(E)} + \frac{b}{2}\sin\theta^{(E)} + \frac{a}{2}\cos\theta^{(E)} \Rightarrow \frac{a}{4} = y_G^{(E)} \tag{13}$$

From the equations of motion, we obtain:

$$\begin{aligned} 0 &= -k\left(x_G^{(E)} - \frac{b}{2}\cos\theta^{(E)} + \frac{a}{2}\sin\theta^{(E)}\right) \Rightarrow x_G^{(E)} = 0 \\ 0 &= -k\left(y_G^{(E)} - \frac{b}{2}\cos\theta^{(E)} - \frac{a}{2}\sin\theta^{(E)}\right) - mg + R \Rightarrow R = mg - \frac{3}{4}ka \end{aligned}$$

The Euler equation does not provide with information ($0 = 0$) because neither of the forces produce torque.

(d) The linearization of the Euler equation around the equilibrium position (assuming the position of the center of mass remains fixed) determines the small oscillations of the rectangle. The linearized equation turns out to be:

$$I_G\ddot{\phi} = \frac{k}{2}\left[by_G^{(E)}(-\phi\sin\theta^{(E)}) - ay_G^{(E)}\cos\theta^{(E)}\phi\right] + R\left[\frac{b}{2}(-\phi\sin\theta^{(E)} - \frac{a}{2}\phi\cos\theta^{(E)})\right] \tag{14}$$

$$I_G\ddot{\phi} = -\left(\frac{ka}{2} + mg\right)a\phi \tag{15}$$

Therefore, the frequency of oscillation is:

$$\omega = \frac{a}{I_G}\frac{ka + 2mg}{2} \tag{16}$$