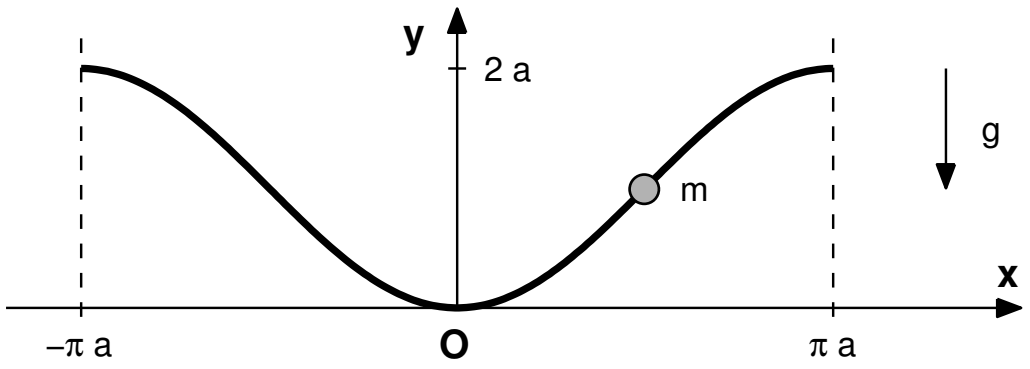
1. A heavy point particle P of mass m is constrained to move without friction along the curve

$$y = a\left(1 - \cos\left(\frac{x}{a}\right)\right), \tag{1}$$

in the range $-\pi a \leq x \leq \pi a$, where a is a positive constant. Initially, the particle is located at $x = 0$ with velocity v_0 along the positive x axis.



- Reduce the problem to quadratures.
- If $v_0 = \sqrt{ga}$ determine the minimum and maximum value of x along the motion.
- What should be the minimum value of v_0 in order to reach $x = \pi a$? How much time would it take to reach that point?

Let's assume now that the curve containing the point particle rotates around the OY axis with constant angular velocity $\boldsymbol{\omega} = \Omega \boldsymbol{j}$.

- Define a convenient moving reference frame with origin in O, $OX_1Y_1Z_1$. Determine the inertia force that acts on the particle in this moving reference frame as a function of m,Ω,x_1 and $\dot{x}_1$, where x_1 represents the position coordinate of P along OX_1.
- Reduce the problem to quadratures and determine the value of the reaction force.
- Find out a trascendental equation whose solutions are the equilibrium positions of the point particle in $OX_1Y_1Z_1$.
- If Ω were a function of time, would it be possible to use the energy equation? Justify your answer.

- The point particle P moves in a plane. This corresponds to two degrees of freedom. The particle is constrained to move along a curve. This constraint removes one of the degrees of freedom but adds a reaction force $\boldsymbol{R}$ along the normal direction to the curve. The position of the particle is completely specified by the coordinate x.

  The particle is subject to two forces: its own weight and the reaction. The reaction does not perform work and the weight derives from a potential

$$V = mgy = mga\left(1 - \cos\left(\frac{x}{a}\right)\right), \tag{2}$$

  therefore, the energy is conserved. Since there is only one degree of freedom and the energy is conserved, it is possible to use the equation of conservation of energy, $T + V = E$, to solve the problem. The kinetic energy can be expressed as

$$T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) = \frac{1}{2}m\dot{x}^2\left(1 + \sin^2\left(\frac{x}{a}\right)\right), \tag{3}$$

  and the total energy E is obtained imposing the initial condition $E = (1/2)mv_0^2$. Thus,

$$\dot{x}^2\left(1 + \sin^2\left(\frac{x}{a}\right)\right) + 2ga\left(1 - \cos\left(\frac{x}{a}\right)\right) = v_0^2. \tag{4}$$

  Defining $G_1(x)$ as

$$G_1(x) = \frac{v_0^2 - 2ga\left(1 - \cos\left(\frac{x}{a}\right)\right)}{\left(1 + \sin^2\left(\frac{x}{a}\right)\right)}, \tag{5}$$

  equation (4) leads to:

$$\dot{x}^2 = G_1(x) \rightarrow t = \int_0^x \frac{\pm \mathrm{d}x}{\sqrt{G_1(x)}}. \tag{6}$$

- The maximum and minimum values of x are obtained when $\dot{x}$ is 0. From equation (4) this happens when

$$v_0^2 - 2ga\left(1 - \cos\left(\frac{x}{a}\right)\right) = 0, \tag{7}$$

  thus

$$\cos\left(\frac{x}{a}\right) = 1 - \frac{v_0^2}{2ga}. \tag{8}$$

  Imposing $v_0 = \sqrt{ga}$, it leads to $-\frac{\pi}{3}a \leq x \leq \frac{\pi}{3}a$.

- This can be obtained by imposing $x = \pi a$ in equation (8). The result is $v_0 = 2\sqrt{ga}$. If $v_0 = 2\sqrt{ga}$, $G_1(\pi a) = 0$ and from equation (6), it is possible to infer that the particle will reach $x = \pi a$ when $t \rightarrow \infty$.

- We define the moving reference frame such that 1) the axis OY_1 coincides at all times with OY 2) the axis OX_1 is contained in the plane OXZ and rotates at a constant angular velocity $\boldsymbol{\omega} = \Omega \boldsymbol{j}$ and 3) the axis OZ_1 is orthogonal to OX_1 and OY_1 forming a righ-handed coordinate system. In this coordinate system the relative position of P is given by

$$\boldsymbol{r_1} = x_1\boldsymbol{i}_1 + y_1\boldsymbol{j}_1 = x_1\boldsymbol{i}_1 + a\left(1 - \cos\left(\frac{x_1}{a}\right)\right)\boldsymbol{j}_1 \tag{9}$$

  the relative velocity is

$$\boldsymbol{v_1} = \dot{x}_1\left(\boldsymbol{i}_1 + \sin\left(\frac{x_1}{a}\right)\boldsymbol{j}_1\right) \tag{10}$$

and the relative acceleration

$$\boldsymbol{a_1} = \ddot{x}_1 \left(\boldsymbol{i}_1 + \sin\left(\frac{x_1}{a}\right) \boldsymbol{j}_1 \right) + \frac{1}{a}\dot{x}_1^2 \cos\left(\frac{x_1}{a}\right) \boldsymbol{j}_1. \tag{11}$$

Only two of the inertia force terms are non-zero: the centrifugal force and the Coriolis force. The centrifugal force can be expressed

$$\boldsymbol{F_{centr}} = -m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r_1}) = m\Omega^2 x_1 \boldsymbol{i}_1, \tag{12}$$

and the Coriolis force

$$\boldsymbol{F_{cor}} = -2m\boldsymbol{\omega} \times \boldsymbol{v_1} = 2m\Omega\dot{x}_1 \boldsymbol{k}_1. \tag{13}$$

- As before, there is just one degree of freedom, in this case x_1. The weight and the centrifugal force derive from a potential

$$V = mga\left(1 - \cos\left(\frac{x_1}{a}\right)\right) - \frac{1}{2}m\Omega^2 x_1^2. \tag{14}$$

The reaction force has now two components

$$\boldsymbol{R} = R_1 \boldsymbol{n_1} + R_2 \boldsymbol{k}_1, \tag{15}$$

where $\boldsymbol{n_1}$ is a unitary vector normal to the curve within the plane OX_1Y_1. The reaction force does not perform work.

Finally the Coriolis force is always perpendicular to the plane where the relative motion is contained, it does not perform work. Again the problem can be solved using $T + V = E$

$$\dot{x_1}^2 \left(1 + \sin^2\left(\frac{x_1}{a}\right)\right) + 2ga\left(1 - \cos\left(\frac{x_1}{a}\right)\right) - \Omega^2 x_1^2 = v_0^2. \tag{16}$$

Defining $G_2(x_1)$ as

$$G_2(x_1) = \frac{v_0^2 - 2ga\left(1 - \cos\left(\frac{x_1}{a}\right)\right) + \Omega^2 x_1^2}{\left(1 + \sin^2\left(\frac{x_1}{a}\right)\right)}, \tag{17}$$

equation (16) leads to:

$$\dot{x}_1^2 = G_2(x_1) \rightarrow t = \int_0^{x_1} \frac{\pm \mathrm{d}x_1}{\sqrt{G_2(x_1)}}. \tag{18}$$

In order to obtain the reaction force, we need to use relative dynamics.

$$\sum \boldsymbol{F} + \sum \boldsymbol{F_I} = m\boldsymbol{a_1}, \tag{19}$$

where $\sum \boldsymbol{F}$ is the sum of the external forces and $\sum \boldsymbol{F_I}$ is the sum of the inertia forces. We assume that we already know $x_1(t)$ by integration of equation (18). We need to project the equations along $\boldsymbol{n_1}$ and $\boldsymbol{k}_1$ to obtain R_1 and R_2. The second one is simple since $\boldsymbol{a_1}$ does not have a component along $\boldsymbol{k}_1$ and the only force with a component along $\boldsymbol{k}_1$ is the Coriolis force. Therefore,

$$R_2 = -2m\Omega\dot{x}_1. \tag{20}$$

A unitary normal vector to the curve $\boldsymbol{n_1}$ might be defined as

$$\boldsymbol{n_1} = \frac{1}{\sqrt{1 + (dy/dx)^2}} \left(-\frac{dy}{dx}\boldsymbol{i}_1 + \boldsymbol{j}_1 \right) = \frac{1}{\sqrt{1 + (\sin(x/a))^2}} \left(-\sin\left(\frac{x}{a}\right) \boldsymbol{i}_1 + \boldsymbol{j}_1 \right). \tag{21}$$

The projections of the forces and the acceleration along $\boldsymbol{n_1}$ are:

$$\boldsymbol{W}\cdot\boldsymbol{n_1} = \frac{-mg}{\sqrt{1+(\sin(x_1/a))^2}} \tag{22}$$

$$\boldsymbol{F_{centr}}\cdot\boldsymbol{n_1} = \frac{-m\Omega^2 x_1\sin(x_1/a)}{\sqrt{1+(\sin(x_1/a))^2}} \tag{23}$$

$$\boldsymbol{a_1}\cdot\boldsymbol{n_1} = \frac{(\dot{x}_1^2/a)\cos(x_1/a)}{\sqrt{1+(\sin(x_1/a))^2}} \tag{24}$$

and we obtain

$$R_1 = \frac{mg + m\Omega^2 x_1\sin(x_1/a) + (m\dot{x}_1^2/a)\cos(x_1/a)}{\sqrt{1+(\sin(x_1/a))^2}}. \tag{25}$$

- For the equilibrium position the relative acceleration needs to be zero, e.g.

$$\sum \boldsymbol{F} + \sum \boldsymbol{F_I} = 0. \tag{26}$$

If we project equation (26) along $\boldsymbol{n_1}$ and $\boldsymbol{k}_1$ we will obtain the reaction forces in the equilibrium position. Therefore, we can ignore these two projections. We need to project along the tangent direction. This time we do not need to use a unitary vector, so we can compute the scalar product between equation (26) and the vector

$$\boldsymbol{t_1} = \boldsymbol{i}_1 + \frac{dy}{dx}\boldsymbol{j}_1 = \boldsymbol{i}_1 + \sin\left(\frac{x}{a}\right)\boldsymbol{j}_1. \tag{27}$$

This leads to

$$\boldsymbol{W}\cdot\boldsymbol{t_1} = -mg\sin(x_1/a) \tag{28}$$

$$\boldsymbol{F_{centr}}\cdot\boldsymbol{t_1} = m\Omega^2 x_1, \tag{29}$$

since obviously the reaction force does not have a component along $\boldsymbol{t_1}$. Therefore, the trascendental equation that provides the equilibrium solution is

$$\frac{\Omega^2 a}{g}\frac{x_1}{a} = \sin\left(\frac{x_1}{a}\right). \tag{30}$$

Note that $x_1 = 0$ is always an equilibrium position. Two additional solutions might exist depending on the numerical value of the ratio $\frac{\Omega^2 a}{g}$. If $\frac{\Omega^2 a}{g} \geq 1$ no additional solutions exist. It is easy to see this for small angles, by looking at the Taylor series expansion of the function $\sin(x)$

$$\sin(x) = x - \frac{1}{6}x^3 + ... \tag{31}$$

If $\frac{\Omega^2 a}{g} = 1 + \varepsilon$ with $\varepsilon \ll 1$, introducing the expression (31) into equation (30) we obtain

$$6\varepsilon x = -x^3 \tag{32}$$

an equation with three real solutions if $\varepsilon < 0$ and with one real and two complex solutions if $\varepsilon > 0$.

- If Ω is a function of time then the centrifugal force is a function of the position and time and does not fulfill the condition to derive from a potential (to be a function of the position only). Therefore, it is not possible to use the energy equation to solve the problem. Note that the additional term that appears in the inertia force is not a problem since $\mathrm{d}\boldsymbol{\omega}/\mathrm{d}t \times \boldsymbol{r_1}$ is parallel to $\boldsymbol{k}_1$ and does not perform work.

2. The Upper Atmosphere Research Satellite (UARS) was a NASA-operated orbital observatory whose mission was to study the Earth's atmosphere, particularly the protective ozone layer. After concluding its operative life, UARS became an orbital debris. Due to the size of the satellite, the estimation of the location of the re-entry was crucial and a highly-accurate orbit determination was required.

   Let's assume that, at this stage, a radar measure is taken when the satellite is directly overhead the station: at that instant of time, the distance of the satellite to the station is equal to ρ and the range-rate, that is, the variation of the distance, is $\dot{\rho} = 0$. It is also known that the period of the orbit is:

$$\tau = 2\pi \frac{(3(R_\oplus + \rho))^{3/2}}{2\sqrt{2\mu}}$$

   where μ is the gravitational constant of the Earth, and $R_\oplus$ is the Earth's radius.

   (a) Compute as a function of ρ and the parameters μ, $R_\oplus$, the angular velocity of the Earth rotation, $\Omega_\oplus$, and the latitude of the station, λ:
   - Semi-major axis of the orbit.
   - Radius of perigee and apogee.
   - Modulus of the relative velocity of the satellite when the measurement is taken, assuming that the relative velocity of the satellite is directed along the East.

   (b) The effect of the aerodinamic drag over half an orbit is modeled as a Hohmann transfer. The first impulse is applied at the location directly overhead the radar station. After the transfer, the satellite will follow a circular orbit of radius $R_\oplus + \rho/2$ (assuming $\rho < 4R_\oplus$). Compute the value of the required total Δv as a function of ρ, μ and $R_\oplus$.

(a) The semi-major axis of the orbit can be computed from the orbital period:

$$\tau^2 = \frac{4\pi^2}{\mu} a^3 \Rightarrow a = \frac{\tau^{2/3}\mu^{1/3}}{(2\pi)^{2/3}} = \frac{3}{2}(R_\oplus + \rho) \tag{33}$$

The measurement of the range-rate indicates that the radial velocity of the satellite is zero. This fact takes place on the apo or periapsis of the orbit (for elliptical orbits). Therefore, the position overhead the station is either the perigee or the apogee of the UARS. Given that the semi-major axis is greater than the actual radius at this position, it must be the perigee. Then:

$$r_p = (R_\oplus + \rho) \qquad r_a = 2a - r_p = 2(R_\oplus + \rho)$$

The same conclusion can be drawn from the computation of the relative position and velocity. Let $\boldsymbol{r}_0$ and $\boldsymbol{v}_0$ be the position and velocity of the radar station at the measurement time. Their values will be: $\boldsymbol{r}_0 = R_\oplus \boldsymbol{u}_r$ while $\boldsymbol{v}_0 = \Omega R_\oplus \cos\lambda \boldsymbol{u}_E$, where $\boldsymbol{u}_E$ is a unitary vector along the local East direction. The absolute position of the satellite will be:

$$\boldsymbol{r}_s = \boldsymbol{r}_0 + \boldsymbol{r}^{'} = (R_\oplus + \rho)\boldsymbol{u}_r$$

In turn, the absolute velocity can be computed as:

$$\boldsymbol{v}_s = \boldsymbol{v}_0 + \boldsymbol{v}^{'} + \Omega \boldsymbol{k} \times \boldsymbol{r}^{'}$$

where $\boldsymbol{k}$ is along the earth axis of rotation. The relative velocity $\boldsymbol{v}^{'}$ can be computed by differenciation:

$$\boldsymbol{v}^{'} = \frac{d\boldsymbol{r}^{'}}{dt} = \dot{\rho}\boldsymbol{u}_r + \rho\dot{\theta}\boldsymbol{u}_\theta = \rho\dot{\theta}\boldsymbol{u}_\theta$$

In the previous expression both $\dot{\theta}$ and $\boldsymbol{u}_\theta$ are unknowns. In particular, $\boldsymbol{u}_\theta$ may be or may be not equal to $\boldsymbol{u}_E$. The absolute velocity of the satellite is, therefore,

$$\boldsymbol{v}_s = \Omega(R_\oplus + \rho)\cos\lambda \boldsymbol{u}_E + \rho\dot{\theta}\boldsymbol{u}_\theta$$

When the measurement is taken, the condition $\boldsymbol{r}_s \cdot \boldsymbol{v}_s = 0$ is fulfilled. This condition only occurs at the apo or periapsis of the orbit.

To solve for the modulus of the relative velocity, we need to obtain $\dot{\theta}$ as a function of the parameters of the problem. The specific energy equation can be used to this purpose.

$$\frac{1}{2}v_s^2 - \frac{\mu}{r_S} = -\frac{\mu}{2a}$$

Substituting the prevoius results, we obtain,

$$v_S^2 = \frac{4}{3}\frac{\mu}{R_\oplus + \rho}$$

Given that the relative velocity is directed along the East direction, $\boldsymbol{u}_\theta = \boldsymbol{u}_E$ and $\boldsymbol{v}_S = (\Omega(R_\oplus + \rho)\cos\lambda + \rho\dot{\theta})\boldsymbol{u}_E$. The value of $\dot{\theta}$ as a function of the parameters of the problem turn out to be:

$$\dot{\theta} = \frac{1}{\rho}\left[\sqrt{\frac{4}{3}\frac{\mu}{R_\oplus + \rho}} - \Omega(R_\oplus + \rho)\cos\lambda\right]$$

And the modulus of the angular velocity is:

$$v^{'} = \sqrt{\frac{4}{3}\frac{\mu}{R_\oplus + \rho}} - \Omega(R_\oplus + \rho)\cos\lambda$$

(b) To compute the total Δv, we must consider the two impulses of the maneuver. The notation of each of the orbits involved will be: superscript (O) indicates the original orbit, superscript (T) indicates the transfer orbit and superscript (F) indicates the final orbit.

The first impulse is in charge of entering the transfer orbit at the perigee of the original orbit. The semi-major axis of the transfer orbit is: $a^{(T)} = (R_\oplus + \rho + R_\oplus + \rho/2)/2 = R_\oplus + 3\rho/4$. The needed Δv required to perform such maneuvre is computed as the difference between the transfer orbit velocity and the original orbit velocity at the maneuver position.

$$v^{(O)} = \sqrt{\frac{2\mu}{R_\oplus + \rho} - \frac{2\mu}{3(R_\oplus + \rho)}} = \sqrt{\frac{4\mu}{3(R_\oplus + \rho)}}$$

$$v_1^{(T)} = \sqrt{\frac{2\mu}{R_\oplus + \rho} - \frac{\mu}{R_\oplus + 3\rho/4)}} = \sqrt{\frac{\mu(R_\oplus + \rho/2)}{(R_\oplus + \rho)(R_\oplus + 3\rho/4)}}$$

And $(\Delta v)_1 = \left| v_1^{(T)} - v^{(O)} \right|$. Note that $v^{(T)} > v^{(O)}$ iff $\rho < 4R_\oplus$

In the same way we can compute the Δv of the second maneuver.

$$v_2^{(T)} = \sqrt{\frac{2\mu}{R_\oplus + \rho/2} - \frac{\mu}{(R_\oplus + 3\rho/4)}} = \sqrt{\frac{\mu(R_\oplus + \rho)}{(R_\oplus + \rho/2)(R_\oplus + 3\rho/4)}}$$

$$v^{(F)} = \sqrt{\frac{\mu}{R_\oplus + \rho/2}}$$

And $(\Delta v)_2 = \left| v^{(F)} - v_2^{(T)} \right|$. Finally, $\Delta v_T = (\Delta v)_1 + (\Delta v)_2$

3. The motion of a blade of a helicopter is to be analyzed. The blade is assumed to be a homogeneous flat plate of mass m, length ℓ and width b. The tensor of inertia of the blade with respect to its center of mass expressed in the principal axes is considered to be known and equal to:

$$\bar{\bar{I}}_G = \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix}$$

where $I_{xx} = mb^2/12$, $I_{yy} = m\ell^2/12$ and $I_{zz} = m(\ell^2 + b^2)/12$.

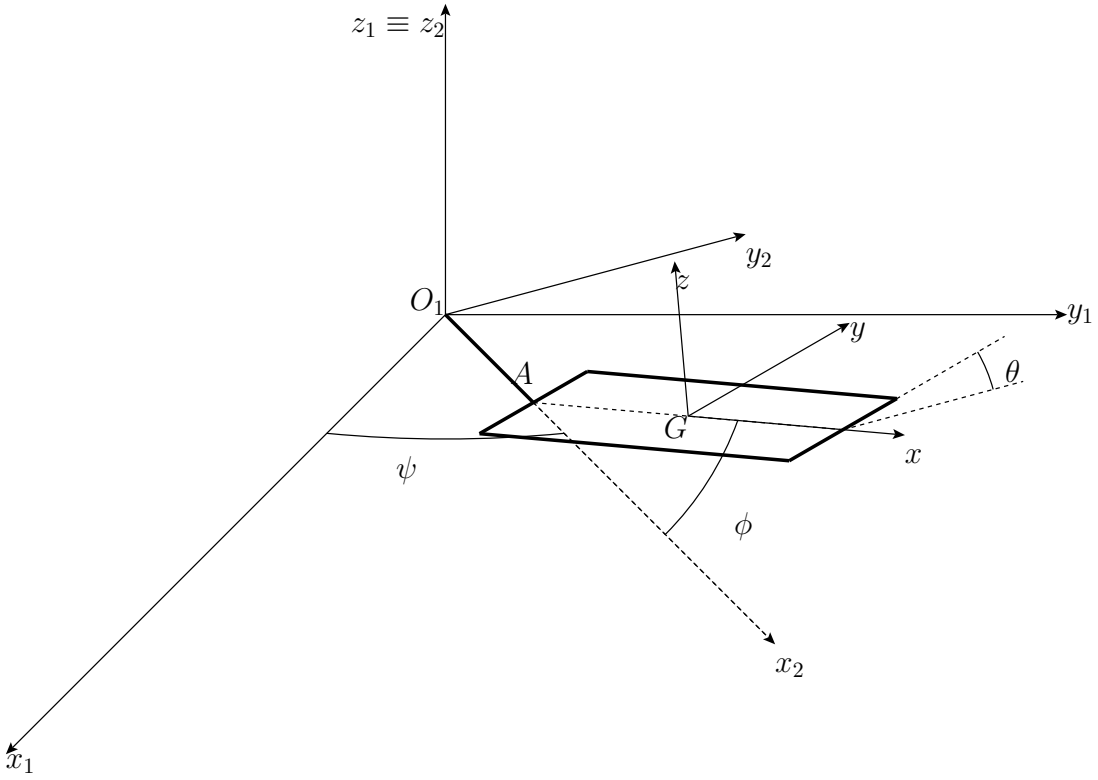
The blade is joint to the rotor of the helicopter by a massless bar of length b. This bar is constrained to move in a plane perpendicular to the rotor axis, O_1z_1, with a constant angular velocity Ω. The junction between the bar and the blade is such that the side BC of the blade is always contained in a plane perpendicular to the bar O_1A.

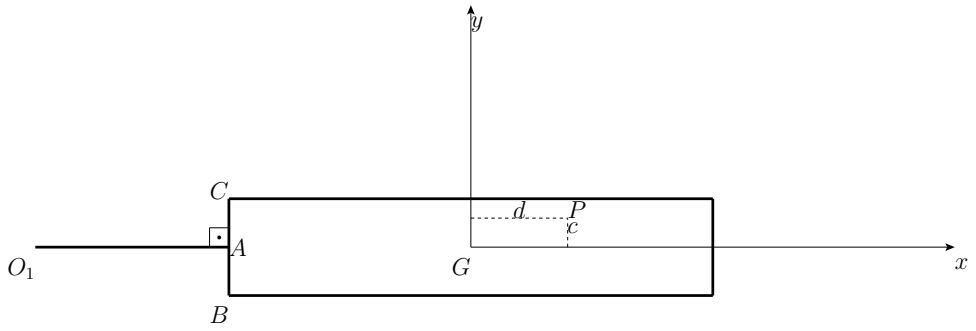
Therefore, the position of the system is completely specified by three angles: ψ, the blade azimuth angle (considering the constraint of the bar, $\psi = \Omega t + \psi_0$); ϕ, the blade flapping angle; and θ, the blade pitch angle, as shown in the figure.

The blade is subject to the effects of gravity and aerodynamic forces. The resultant of the aerodynamic forces is a known force $L(t)$ acting at the center of pressure P and always normal to the plane of the blade. For the sake of simplicity, the position of the center of pressure P is assumed to be time independent. The offset of P with respect to the center of mass of the blade, G, is d in the longitudinal direction and c in the transversal direction, as shown in the figure.

Two reference frames may be used in this analysis: an inertial reference frame: $O_1x_1y_1z_1$ and a body-fixed reference frame $Gxyz$. For better understanding, an additional reference frame $O_1x_2y_2z_2$ has been included in the figure below.

- Express the angular velocity and angular acceleration of the blade in their components with respect to $Gxyz$.
- Compute the tensor of inertia and the angular momentum of the blade with respect to the fixed point O_1.
- Compute the torque of the external forces with respect to the origin O_1 expressing each of them in the most convenient reference frame.
- Derive the equations of motion for ϕ and θ.
- Indicate the equation that would allow us to compute the required applied torque.


$z_1 \equiv z_2$
y_2
z
O_1
y_1
y
A
θ
G
x
ψ
ϕ
x_2
x_1


y
C
d
P
c
A
O_1
G
x
B

(a) The angular velocity of the blade can be expressed as:

$$\boldsymbol{\omega} = \dot{\psi}\boldsymbol{k}_1 + \dot{\phi}\boldsymbol{j}_2 + \dot{\theta}\boldsymbol{i}$$

In order to express the angular velocity in terms of its components in the reference frame $Gxyz$, we must transform $\boldsymbol{k}_1$ and $\boldsymbol{j}_2$ into their xyz components. Given that the transformations are going to be needed throughout the problem, a compilation of the required relationships is listed below:

$$\mathcal{R}_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & \sin(\theta) \\ 0 & -\sin(\theta) & \cos(\theta) \end{bmatrix} \quad \mathcal{R}_y(\phi) = \begin{bmatrix} \cos(\phi) & 0 & \sin(\phi) \\ 0 & 1 & 0 \\ -\sin(\phi) & 0 & \cos(\phi) \end{bmatrix} \mathcal{R}_z(\psi) = \begin{bmatrix} \cos(\psi) & \sin(\psi) & 0 \\ -\sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

In a compact form, the transformation from body-fixed axes to inertial axes is written as:

$$\begin{Bmatrix} \boldsymbol{i} \\ \boldsymbol{j} \\ \boldsymbol{k} \end{Bmatrix} = \mathcal{R} \begin{Bmatrix} \boldsymbol{i}_1 \\ \boldsymbol{j}_1 \\ \boldsymbol{k}_1 \end{Bmatrix} \qquad \mathcal{R} = \mathcal{R}_x(\theta)\mathcal{R}_y(\phi)\mathcal{R}_z(\psi)$$

For completeness, the expression of $\mathcal{R}$ is gathered here:

$$\mathcal{R} = \begin{bmatrix} \cos(\phi)\cos(\psi) & \sin(\psi)\cos(\phi) & \sin(\phi) \\ -\sin(\phi)\sin(\theta)\cos(\psi) - \sin(\psi)\cos(\theta) & -\sin(\phi)\sin(\psi)\sin(\theta) + \cos(\psi)\cos(\theta) & \sin(\theta)\cos(\phi) \\ -\sin(\phi)\cos(\psi)\cos(\theta) + \sin(\psi)\sin(\theta) & -\sin(\phi)\sin(\psi)\cos(\theta) - \sin(\theta)\cos(\psi) & \cos(\phi)\cos(\theta) \end{bmatrix}$$

Therefore, $\boldsymbol{k}_1$ expressed in the body-fixed reference frame is the third row of $\mathcal{R}^T$:

$$\boldsymbol{k}_1 = \sin(\phi)\,\boldsymbol{i} + \sin(\theta)\cos(\phi)\,\boldsymbol{j} + \cos(\phi)\cos(\theta)\,\boldsymbol{k}$$

In turn, $\boldsymbol{j_2} = \cos(\theta)\boldsymbol{j} - \sin(\theta)\boldsymbol{k}$. Thus, the resulting angular velocity is:

$$\boldsymbol{\omega} = \omega_x\boldsymbol{i} + \omega_y\boldsymbol{j} + \omega_z\boldsymbol{k} = \Big(\dot{\theta} + \dot{\psi}\sin(\phi)\Big)\,\boldsymbol{i} + \Big(\dot{\phi}\cos(\theta) + \dot{\psi}\sin(\theta)\cos(\phi)\Big)\,\boldsymbol{j} + \Big(\dot{\psi}\cos(\phi)\cos(\theta) - \dot{\phi}\sin(\theta)\Big)\,\boldsymbol{k}$$

Concerning the angular acceleration, it can be computed differenciating the previous expression with respect to time in the inertial reference frame. It is noted that $\dot{\psi}$ is constant and equal to Ω, and therefore, $\ddot{\psi} = 0$.

$$\boldsymbol{\alpha} = \left.\frac{d\boldsymbol{\omega}}{dt}\right|_{Gxyz} + \boldsymbol{\omega}\times\boldsymbol{\omega} = \Big(\ddot{\theta} + \Omega\dot{\phi}\cos(\phi)\Big)\,\boldsymbol{i} + \Big(\ddot{\phi}\cos(\theta) - \dot{\phi}\dot{\theta}\sin(\theta) + \Omega\dot{\theta}\cos(\theta)\cos(\phi) - \Omega\dot{\phi}\sin(\theta)\sin(\phi)\Big)\,\boldsymbol{j} +$$
$$+ \Big(-\Omega(\dot{\phi}\sin(\phi)\cos(\theta) + \dot{\theta}\cos(\phi)\sin(\theta)) - \ddot{\phi}\sin(\theta) - \dot{\phi}\dot{\theta}\cos(\theta)\Big)\,\boldsymbol{k}$$

(b) To compute the angular momentum with respect the point O_1, we will use: $\boldsymbol{H}_{O_1} = \bar{\bar{I}}_{O_1}\cdot\boldsymbol{\omega}$, since O_1 is a fixed point of the system of rigid bodies. Then, we first have to compute the moment of inertia with respect to the origin. Using the Steiner theorem:

$$\bar{\bar{I}}_{O_1} = \bar{\bar{I}}_G + m\left[\boldsymbol{r}_G^2\bar{\bar{U}} - [\boldsymbol{r}_G\boldsymbol{r}_G]\right]$$

The position of the center of mass of the blade is: $\boldsymbol{r}_G = b\boldsymbol{i}_2 + \ell/2\boldsymbol{i}$. We can express this vector in the inertial reference frame:

$$\boldsymbol{r}_G = (b + \ell/2\cos\phi)\cos\psi\boldsymbol{i}_1 + (b + \ell/2\cos\phi)\sin\psi\boldsymbol{j}_1 + \ell/2\sin\phi\boldsymbol{k}_1$$

or in terms of its xyz components, the position of the center of mass is:

$$\boldsymbol{r}_G = (b\cos\phi + \ell/2)\,\boldsymbol{i} - b\sin\phi\,(\sin(\theta)\boldsymbol{j} + \cos(\theta)\boldsymbol{k})$$

Therefore, the final expression of the moment of inertia will be:

$$\bar{\bar{I}}_{O_1} = \begin{bmatrix} I_{xx}^{'} & \lambda_1 & \lambda_2 \\ \lambda_1 & I_{yy}^{'} & \lambda_3 \\ \lambda_2 & \lambda_3 & I_{zz}^{'} \end{bmatrix}$$

where

$$\begin{aligned}
\lambda_1 &= mb\sin\phi\cos(\theta)(b\cos\phi + \ell/2) \\
\lambda_2 &= mb\sin\phi\sin(\theta)(b\cos\phi + \ell/2) \\
\lambda_3 &= -mb^2\sin^2\phi\cos(\theta)\sin(\theta) \\
I_{xx}^{'} &= I_{xx} + mb^2\sin^2(\phi) \\
I_{yy}^{'} &= I_{yy} + m\left[b^2\sin^2(\phi)\cos^2(\theta) + (b\cos(\phi) + \ell/2)^2\right] \\
I_{zz}^{'} &= I_{zz} + m\left[b^2\sin^2(\phi)\sin^2(\theta) + (b\cos(\phi) + \ell/2)^2\right]
\end{aligned}$$

Thus, the angular momentum is:

$$\begin{aligned}
\boldsymbol{H}_{O_1} &= \left(I_{xx}^{'}\omega_x + \lambda_1\omega_y + \lambda_2\omega_z\right)\boldsymbol{i} + \left(\lambda_1\omega_x + I_{yy}^{'}\omega_y + \lambda_3\omega_z\right)\boldsymbol{j} + \left(\lambda_2\omega_x + \lambda_3\omega_y + I_{zz}^{'}\omega_z\right)\boldsymbol{k} = \\
&= H_x^{0_1}\boldsymbol{i} + H_y^{0_1}\boldsymbol{j} + H_z^{0_1}\boldsymbol{k}
\end{aligned}$$

(c) There are two external forces: gravity and the resultant of the aerodynamic pressure. Both torques can be computed as the cross product of the position of the resultant times the force. For the gravity, the easiest way is to use the inertial reference frame. Thus,

$$\boldsymbol{M}_g = \boldsymbol{r}_G \times (-mg\boldsymbol{k}_1) = -mg\,(b + \ell/2\cos\phi)\,(\sin\psi\boldsymbol{i}_1 - \cos\psi\boldsymbol{j}_1)$$

In turn, the torque due to the aerodynamic force is easily computed in terms of the xyz components:

$$\boldsymbol{M}_a = (\boldsymbol{r}_G + \boldsymbol{GP}) \times L(t)\boldsymbol{k} = L(t)\left[(c - b\sin(\phi)\sin(\theta))\,\boldsymbol{i} - (b\cos\phi + \ell/2 + d)\,\boldsymbol{j}\right]$$

(d) The equations of motion in which no reactions appear are the equation of angular momentum in $\boldsymbol{i}_1$ and $\boldsymbol{j}_1$ directions. This means that we can obtain two ODEs of second order for the evolution of ϕ and θ without any other unknown in them. The equations we are searching for are:

$$\begin{aligned}
\left(\frac{d\boldsymbol{H}_{O_1}}{dt} + \boldsymbol{\omega}\times\boldsymbol{H}_{O_1} = \boldsymbol{M}_g + \boldsymbol{M}_a + \Gamma\boldsymbol{k}_1\right) &\cdot \boldsymbol{i}_1 \\
\left(\frac{d\boldsymbol{H}_{O_1}}{dt} + \boldsymbol{\omega}\times\boldsymbol{H}_{O_1} = \boldsymbol{M}_g + \boldsymbol{M}_a + \Gamma\boldsymbol{k}_1\right) &\cdot \boldsymbol{j}_1
\end{aligned}$$

The needed components of the gravity torque are:

$$\begin{aligned}
\boldsymbol{M}_g\cdot\boldsymbol{i}_1 &= -mg\left(b + \frac{\ell}{2}\cos\phi\right)\sin\psi \\
\boldsymbol{M}_g\cdot\boldsymbol{j}_1 &= mg\left(b + \frac{\ell}{2}\cos\phi\right)\cos\psi
\end{aligned}$$

To compute the needed components of the aerodynamic torque, it is requiered to express $\boldsymbol{i}_1$ and $\boldsymbol{j}_1$ in their xyz components.

$$\begin{aligned}\boldsymbol{i}_1 &= \cos(\phi)\cos(\psi)\boldsymbol{i} - (\sin(\psi)\cos(\theta) + \cos(\psi)\sin(\theta)\sin(\phi))\,\boldsymbol{j} + (\sin\psi\sin\theta - \cos(\psi)\cos(\theta)\sin(\phi))\,\boldsymbol{k}\\ \boldsymbol{j}_1 &= \sin(\phi)\cos(\psi)\boldsymbol{i} + (\cos(\psi)\cos(\theta) - \sin(\psi)\sin(\theta)\sin(\phi))\,\boldsymbol{j} - (\cos\psi\sin\theta + \sin(\psi)\cos(\theta)\sin(\phi))\,\boldsymbol{k}\end{aligned}$$

The terms which appear in the equations of motion due to the aerodynamic torque are:

$$\begin{aligned}\boldsymbol{M}_a\cdot\boldsymbol{i}_1 &= L(t)\left[(c - b\sin(\phi)\sin(\theta))\cos\phi\cos\psi + \left(b\cos\phi + \frac{\ell}{2} + d\right)(\sin(\psi)\cos(\theta) + \cos(\psi)\sin(\theta)\sin(\phi))\right]\\ \boldsymbol{M}_a\cdot\boldsymbol{j}_1 &= L(t)\left[(c - b\sin(\phi)\sin(\theta))\sin\phi\cos\psi - \left(b\cos\phi + \frac{\ell}{2} + d\right)(\cos(\psi)\cos(\theta) - \sin(\psi)\sin(\theta)\sin(\phi))\right]\end{aligned}$$

In addition, the result of the cross product of the angular velocity and the angular momentum is:

$$\boldsymbol{\omega}\times\boldsymbol{H}_{O_1} = \left[\omega_y H_z^{O_1} - \omega_z H_y^{O_1}\right]\boldsymbol{i} + \left[\omega_z H_x^{O_1} - \omega_x H_z^{O_1}\right]\boldsymbol{j} + \left[\omega_x H_y^{O_1} - \omega_y H_x^{O_1}\right]\boldsymbol{k}$$

The projection of this vector in the desired directions is:

$$\begin{aligned}(\boldsymbol{\omega}\times\boldsymbol{H}_{O_1})\cdot\boldsymbol{i}_1 &= \cos(\phi)\cos(\psi)\left[\omega_y H_z^{O_1} - \omega_z H_y^{O_1}\right] - (\sin(\psi)\cos(\theta) + \cos(\psi)\sin(\theta)\sin(\phi))\\ &\quad \left[\omega_z H_x^{O_1} - \omega_x H_z^{O_1}\right] + (\sin\psi\sin\theta - \cos(\psi)\cos(\theta)\sin(\phi))\left[\omega_x H_y^{O_1} - \omega_y H_x^{O_1}\right]\\ (\boldsymbol{\omega}\times\boldsymbol{H}_{O_1})\cdot\boldsymbol{j}_1 &= \sin(\phi)\cos(\psi)\left[\omega_y H_z^{O_1} - \omega_z H_y^{O_1}\right] + (\cos(\psi)\cos(\theta) - \sin(\psi)\sin(\theta)\sin(\phi))\\ &\quad \left[\omega_z H_x^{O_1} - \omega_x H_z^{O_1}\right] - (\cos\psi\sin\theta + \sin(\psi)\cos(\theta)\sin(\phi))\left[\omega_x H_y^{O_1} - \omega_y H_x^{O_1}\right]\end{aligned}$$

Finally, the derivative of the angular momentum with respect to time in the auxiliar reference frame is:

$$\frac{d\boldsymbol{H}_{O_1}}{dt} = \frac{dH_x^{0_1}}{dt}\boldsymbol{i} + \frac{dH_y^{0_1}}{dt}\boldsymbol{j} + \frac{dH_z^{0_1}}{dt}\boldsymbol{k}$$

$$\begin{aligned}\frac{dH_x^{0_1}}{dt} &= \frac{dI_{xx}^{'}}{dt}\omega_x + I_{xx}^{'}\frac{d\omega_x}{dt} + \frac{d\lambda_1}{dt}\omega_y + \lambda_1\frac{d\omega_y}{dt} + \frac{d\lambda_2}{dt}\omega_z + \lambda_2\frac{d\omega_z}{dt}\\ \frac{dH_y^{0_1}}{dt} &= \frac{d\lambda_1}{dt}\omega_x + \lambda_1\frac{d\omega_x}{dt} + \frac{dI_{yy}^{'}}{dt}\omega_y + I_{yy}^{'}\frac{d\omega_y}{dt} + \frac{d\lambda_3}{dt}\omega_z + \lambda_3\frac{d\omega_z}{dt}\\ \frac{dH_z^{0_1}}{dt} &= \frac{d\lambda_2}{dt}\omega_x + \lambda_2\frac{d\omega_x}{dt} + \frac{d\lambda_3}{dt}\omega_y + \lambda_3\frac{d\omega_y}{dt} + \frac{dI_{zz}^{'}}{dt}\omega_z + I_{zz}^{'}\frac{d\omega_z}{dt}\end{aligned}$$

The components of the previous expression in the $\boldsymbol{i}_1$ and $\boldsymbol{j}_1$ directions are:

$$\begin{aligned}\frac{d\boldsymbol{H}_{O_1}}{dt}\cdot\boldsymbol{i}_1 &= \cos(\phi)\cos(\psi)\frac{dH_x^{0_1}}{dt} - (\sin(\psi)\cos(\theta) + \cos(\psi)\sin(\theta)\sin(\phi))\frac{dH_y^{0_1}}{dt} +\\ &\quad + (\sin\psi\sin\theta - \cos(\psi)\cos(\theta)\sin(\phi))\frac{dH_z^{0_1}}{dt}\\ \frac{d\boldsymbol{H}_{O_1}}{dt}\cdot\boldsymbol{j}_1 &= \sin(\phi)\cos(\psi)\frac{dH_x^{0_1}}{dt} + (\cos(\psi)\cos(\theta) - \sin(\psi)\sin(\theta)\sin(\phi))\frac{dH_y^{0_1}}{dt} -\\ &\quad - (\cos\psi\sin\theta + \sin(\psi)\cos(\theta)\sin(\phi))\frac{dH_z^{0_1}}{dt}\end{aligned}$$

The equations of motion can be built from the elements computed before.

(e) The equation that allow us to compute the moment of reaction, Γ, is the angular momentum equation in the $\boldsymbol{k}_1$ direction. That is,

$$\left(\frac{d\boldsymbol{H}_{O_1}}{dt} + \boldsymbol{\omega}\times\boldsymbol{H}_{O_1} = \boldsymbol{M}_g + \boldsymbol{M}_a + \Gamma\right)\cdot\boldsymbol{k}_1$$