FLUID MECHANICS I **13-12-2021**
PROBLEM 1 (15 POINTS). **Time: 80 minutes**

NAME:..... **GRADE:.....**

An infinitely long porous cylinder of radius R rotates with angular velocity Ω within a liquid of density ρ and viscosity μ. Through the wall of the cylinder, a uniform volumetric flow rate $q = 2\pi R U_0$ per unit length is injected. Far from the cylinder, the fuid is at rest and the pressure and temperature are p_∞ and T_∞, respectively. Assuming the flow is axially symmetric,

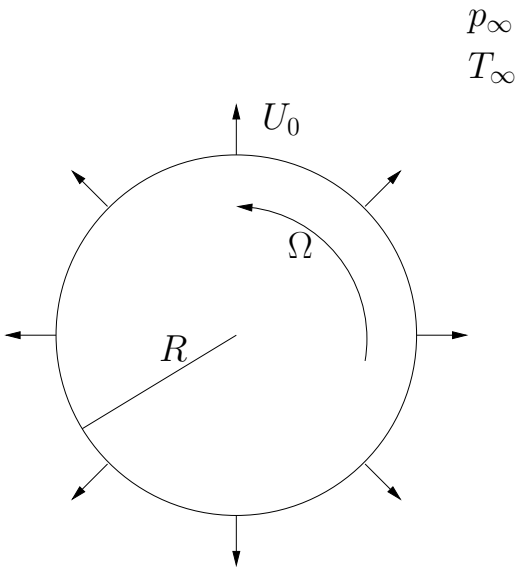
1. Write the ordinary differential equation, with its boundary conditions, that determine the radial component $v_r(r)$ of the velocity. Find $v_r(r)$.

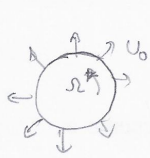
2. Write the ordinary differential equation, with its boundary conditions that determine the tangential component $v_\theta(r)$ of the velocity. Compute $v_\theta(r)$. (hint: look for solutions of the form $v_\theta = Cr^n$, where C and n are constants to be determined as part of the solution.

3. Write the differential equation, with its boundary conditions, that determines the pressure field $p(r)$. Determine $p(r)$.

4. If the cylinder is at a temperature T_0 and the fluid thermal capacity and conductivity are c and k, respectively, write the differential equation, with its boundary conditions, that determine the temperature field.

5. Show that, if Rayleigh's dissipation function ϕ_v is negligible, no stationary solution exists for the temperature field.





① CONTINUITY: $\frac{1}{r}\frac{\partial}{\partial r}(r v_r)=0$; $v_r(R)=U_0$ $\Rightarrow$ $r v_r = C = R U_0$

$$\boxed{v_r = U_0 \frac{R}{r}}$$

② MOMENTUM-θ: $\rho v_r \frac{\partial v_\theta}{\partial r} + \rho \frac{v_r v_\theta}{r} = \mu \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r v_\theta)\right)$

$$\rho U_0 \frac{R}{r}\frac{\partial v_\theta}{\partial r} + \rho U_0 \frac{R}{r}\frac{v_\theta}{r} = \mu \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r v_\theta)\right)$$

$Re = \frac{\rho U_0 R}{\mu}$

$$\frac{Re}{r}\frac{\partial v_\theta}{\partial r} + \frac{Re}{r^2} v_\theta = \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r v_\theta)\right)$$

$r=R$: $v_\theta = \Omega R$

$r\to\infty$: $v_\theta = 0$

$v_\theta = C \cdot r^n$ $\quad C\frac{Re}{r}\cdot n r^{n-1} + \frac{Re}{r^2} C r^n = \frac{\partial}{\partial r}\left(\frac{1}{r}(n+1) C \cdot r^n\right) = C(n+1)(n-1) r^{n-2}$ $\quad \nearrow n=-1$

$Re\, n + Re = (n+1)(n-1) \rightarrow (n+1)Re = (n+1)(n-1) \searrow n = 1+Re$

$v_\theta = A \cdot r^{-1} + B r^{1+Re}$

$r\to\infty$: $0 = \underbrace{A \cdot r^{-1}}_{\frac{A}{R}} + B r^{1+Re} \Rightarrow B = 0$

$r=R$: $\Omega R = \frac{A}{R} \rightarrow A = \Omega R^2$

$$\boxed{v_\theta = \Omega R \frac{R}{r}}$$

③ MOMENTUM-r: $\rho v_r \frac{\partial v_r}{\partial r} - \rho \frac{v_\theta^2}{r} = -\frac{\partial P}{\partial r} + \mu \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r v_r)\right)$ $\quad \leftarrow$ CONTINUITY

$\rho U_0 \frac{R}{r}\cdot\left(-\frac{U_0 R}{r^2}\right) - \rho \frac{\Omega^2 R^2 R^2}{r^3} = -\frac{\partial P}{\partial r}$

$r\to\infty$: $P = P_\infty$

$$\boxed{P - P_\infty = -\frac{1}{2}\rho\left(\frac{R}{r}\right)^2\left(U_0^2 + (\Omega R)^2\right)}$$

④ ENERGY: $\rho c v_r \frac{\partial T}{\partial r} = \phi_v + \frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)$, with $\phi_v = 2\mu\left(\frac{\partial v_r}{\partial r}\right)^2 + 2\mu\frac{v_r^2}{r^2} + \mu\left(r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right)\right)^2$

$\frac{\rho c}{k} U_0 \frac{R}{r}\frac{\partial T}{\partial r} = \frac{1}{k}\phi_v + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)$

$$\boxed{\begin{aligned} &\frac{Pr\, Re}{r}\frac{\partial T}{\partial r} = \frac{1}{k}\phi_v + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) \\ &r=R:\ T=T_0 \\ &r\to\infty:\ T=T_\infty \end{aligned}}$$

$Re = \frac{\rho U R}{\mu}$; $Pr = \frac{\mu c}{k}$

⑤ If $\phi_v \simeq 0 \rightarrow Re\,Pr\frac{dT}{dr} = \frac{d}{dr}\left(r\frac{dT}{dr}\right) \rightarrow$ $r\frac{dT}{dr} - Re\,Pr\,T = A$

$\frac{dT}{A + Re\,Pr\,T} = \frac{dr}{r} \rightarrow \frac{1}{Re\,Pr}\ln(A + Re\,Pr\,T) = \ln r + \ln B$

$T = \frac{B r^{Re\,Pr} - A}{Re\,Pr}$; $r=R$: $T_0 = \frac{B R^{Re\,Pr} - A}{Re\,Pr}$

$T - T_0 = \frac{B(r^{Re\,Pr} - R^{Re\,Pr})}{Re\,Pr}$

$r\to\infty \Rightarrow T - T_0$ GAUSS WITHOUT LIMIT.