① CONTINUITY: $\frac{1}{r}\frac{\partial}{\partial r}(rV_r)=0 \rightarrow rV_r = const = -2RV \Rightarrow \boxed{V_r = -2V\frac{R}{r}}$

at $r=R$: $V_r = \boxed{-2V = V_i}$

IF $V_i \neq -2V$ OTHER TERMS APPEAR IN CONTINUITY $\frac{\partial}{\partial z} \neq 0$

② MOMENTUM-r: $V_r\frac{\partial V_r}{\partial r} = -\frac{\partial P}{\partial r} + \mu\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(rV_r)\right] \rightarrow P + \frac{V_r^2}{2}\rho = f(z)$ ($\frac{\partial}{\partial r}(rV_r) = 0$)

$P = f(z) - \frac{1}{2}\rho\frac{4R^2}{r^2}V^2$

MOMENTUM-z: $\underbrace{\rho V_r\frac{\partial V_z}{\partial r}}_{f_1(r)} = \underbrace{-\frac{\partial P}{\partial z}}_{-f'(z)} + \underbrace{\mu\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V_z}{\partial r}\right)}_{f_2(r)}$ → ONLY POSSIBLE IF $f'(z) = const.$

$\boxed{P = P_0 - P_l z + 2\rho\frac{R^2}{r^2}V^2}$

③ $V_\theta = 0$

$V_r = -2V\frac{R}{r}$

$-2\rho\frac{R}{r}V\frac{dV_z}{dr} = P_l + \frac{\mu}{r}\frac{d}{dr}\left(r\frac{dV_z}{dr}\right) = 0 \rightarrow \mu\frac{d}{dr}\left(r\frac{dV_z}{dr}\right) + 2\rho RV\frac{dV_z}{dr} + P_l \cdot r = 0$

$r=R: V_z = 0$

$r=2R, V_z = 0$

$V_z = V_{zH} + V_{zP}$; $V_{zP} = -\frac{P_l}{4\mu(1+Re)}r^2$; $Re = \frac{\rho VR}{\mu}$

HOMOGENEOUS: $\frac{d}{dr}\left(r\frac{dV_z}{dr}\right) + 2Re\frac{dV_z}{dr} = 0 \rightarrow r\frac{dV_z}{dr} + 2ReV_z = A$

$\frac{dV_z}{A-2ReV_z} = +\frac{dr}{r} \rightarrow -\frac{1}{2Re}\ln(A-2ReV_z) = \ln r + \ln B = \ln(Br)$

RENAMING CONSTANTS: $V_z = A + \frac{B}{r^{2Re}} - \frac{P_l}{4\mu(1+Re)}r^2$

$r=R: V_z = 0 = A + \frac{B}{R^{2Re}} - \frac{P_l}{4\mu(1+Re)}R^2$

$r=2R: V_z = 0 = A + \frac{B}{(2R)^{2Re}} - \frac{P_l}{4\mu(1+Re)}4R^2$

$\boxed{\begin{aligned} B &= -\frac{3P_lR^2}{4\mu(1+Re)}\frac{(2R)^{2Re}}{2^{2Re}-1} \\ A &= \frac{P_lR^2}{4\mu(1+Re)}\frac{2^{2(Re+1)}}{2^{2Re}-1} \\ V_z &= A + \frac{B}{r^{2Re}} - \frac{P_l}{4\mu(1+Re)}r^2 \end{aligned}}$

④



$\frac{d}{dt}\int_{\Sigma_c}\rho\bar{v}\,d\tau + \int_{\Sigma_1}\rho\bar{v}(\bar{v}-\bar{v}_c)\cdot\bar{n}\,d\sigma = \int_{\Sigma_c}-\bar{\bar{T}}\bar{n}\,d\sigma + \int_{\Sigma_2}\bar{\bar{\tau}}\cdot\bar{n}\,d\sigma$

$P_1\pi\left[(2R)^2 - R^2\right]\bar{e}_z - P_2\pi\left[(2R)^2 - R^2\right]\bar{e}_z = \bar{F}_{2R} + \bar{F}_R$

But $P_2 = P_1 + \frac{\partial P}{\partial z}(z_2 - z_1) = P_1 - P_l L$.

$\boxed{\frac{\bar{F}_R + \bar{F}_{2R}}{L} = -P_l \cdot \pi 3R^2}$

$$c_p \rho v_r \frac{\partial T}{\partial r} = \frac{k}{r} \frac{\partial}{\partial r}\left(r \frac{\partial T}{\partial r}\right) \rightarrow \frac{d}{dr}\left(r \frac{dT}{dr}\right) + 2 Re Pr \frac{dT}{dr} = 0$$

$r = R: T = T_R$

$r = 2R: T = T_{2R}$

$$Re = \frac{\rho v R}{\mu} \qquad Pr = \frac{\mu c}{k}$$

$$T = T_R - \frac{(T_{2R} - T_R)\, 2^{2 Re Pr}}{1 - 2^{2 Re Pr}} + \frac{(T_{2R} - T_R)\, 2^{2 Re Pr}}{1 - 2^{2 Re Pr}} \cdot \left(\frac{R}{r}\right)^{2 Re Pr}$$