

(x', y', z') Auxiliary axes

(x, y, z) Fixed reference

C.V: $\Sigma_e, \Sigma_\omega, \Sigma_{S_1}, \Sigma_{S_2}$

① Continuity: $\frac{d}{dt}\int_{V_c} \rho\, dV + \int_{\Sigma_c} \rho(\bar{v}-\bar{v}_c)\cdot\bar{n}\, d\sigma$

$\frac{d}{dt}\int_{V_c}\rho dV \to 0$ Constant mass

Σ_e, Σ_ω, Σ_{S_1}, Σ_{S_2}: $\bar{v}=\bar{v}_c$

$$-\rho Q + 2\rho U_S A_S = 0 \rightarrow \boxed{U_S = \frac{Q}{2A_S}}$$

② MOMENTUM:

$\frac{d}{dt}\int_{V_c}\rho\bar{v}\,dV \to 0$ SYMMETRY

$\int_{\Sigma_c}\rho\bar{v}(\bar{v}-\bar{v}_c)\cdot\bar{n}\,d\sigma$: Σ_e, Σ_ω, Σ_{S_1}, Σ_{S_2}: $\bar{v}=\bar{v}_c$; Σ_{S_1} SYMMETRY

$\int_{\Sigma_c}(-p\bar{n})\,d\sigma + \int_{\Sigma_c}\bar{\bar{\tau}}\cdot\bar{n}\,d\sigma + \int_{V_c}\rho\bar{g}\,dV$: Σ_e manometric pressure; Σ_ω manometric pressure; Σ_{S_1}, Σ_{S_2} manometric pressure; $\bar{v}_c$ UNIFORM; $\Sigma_\omega \to -\bar{F}$; $\bar{v}$ UNIFORM; $\bar{v}$ UNIFORM

$$-\rho\frac{Q}{A}Q\bar{e}_z = (p_e - p_a)A\bar{e}_z - \bar{F} + \rho g V \bar{e}_z$$

$$\boxed{\bar{F} = \left[\rho\frac{Q^2}{A} + (p_e - p_a)A + \rho g V\right]\bar{e}_z}$$

③ ANGULAR MOMENTUM ABOUT O

$\frac{d}{dt}\int_{V_c}(\bar{x}-\bar{x}_0)\times\rho\bar{v}\,dV$ (*) $+ \int_{\Sigma_c}(\bar{x}-\bar{x}_0)\times(\rho\bar{v})(\bar{v}-\bar{v}_c)\cdot\bar{n}\,d\sigma = \int_{\Sigma_c}(\bar{x}-\bar{x}_0)\times(-p\bar{n})\,d\sigma + \int_{\Sigma_c}(\bar{x}-\bar{x}_0)\times(\bar{\bar{\tau}}\cdot\bar{n})\,d\sigma + \int_{V_c}(\bar{x}-\bar{x}_0)\times\rho\bar{g}\,dV$

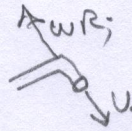
$\frac{d}{dt}\int_{V_c}\rho dV \to 0$ CONSTANT VECTOR

$\int_{\Sigma_c}$: Σ_e SYMMETRY; Σ_ω, Σ_{S_1}, Σ_{S_2}: $\bar{v}=\bar{v}_c$

$\int_{\Sigma_c}$: Σ_e SYMMETRY; Σ_ω; $\Sigma_{S_1} = -\bar{M}_A$; Σ_{S_2}: $p=p_a$

$\int_{V_c}$: $\bar{v}$ UNIFORM; $\bar{v}$ UNIFORM; $\bar{v}$ UNIFORM

$\int_{V_c}$ SYMMETRY



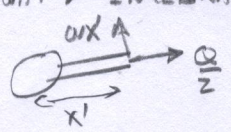
at Σ_1: $\bar{v} = (\omega R - U_S)\bar{e}_{y'}$

$$\underbrace{\rho\frac{Q}{2}(\omega R - U_S)R\bar{e}_z}_{\Sigma_{S_1}} + \underbrace{\rho\frac{Q}{2}(\omega R - U_S)R\bar{e}_z}_{\Sigma_{S_2}} = -\bar{M}$$

$$\boxed{\bar{M} = \rho Q R\left(\frac{Q}{2A} - \omega R\right)\bar{e}_z}$$

④ At equilibrium, no friction $\Rightarrow \bar{M} = 0 \Rightarrow$ $\boxed{\omega = \frac{Q}{2AR}}$

⑤ MODIFY INTEGRAL (*), ALONG THE ARMS:



$\int_{V_c}(\bar{x}-\bar{x}_0)\times\rho\bar{v}\,dV = 2\int_0^R \rho\omega x'\cdot x' A_S dx' \bar{e}_\theta = \frac{2}{3}\rho\omega R^3 A_S \bar{e}_\theta$

$$\bar{M} = \left[\rho Q R\left(\frac{Q}{2A} - \omega R\right) - \frac{2}{3}\rho\dot{\omega}R^3 A_S\right]\bar{e}_z$$

$$I\frac{d\omega}{dt} = \rho Q R\left(\frac{Q}{2A} - \omega R\right) - \frac{2}{3}\rho\dot{\omega}R^3 A_S$$

$$\underbrace{\left[I + \frac{2}{3}\rho R^3 A_S\right]}_{I'}\frac{d\omega}{dt} = \rho Q R\left(\frac{Q}{2A} - \omega R\right), \quad \omega(0)=0$$

$$\boxed{\omega = \frac{Q}{2AR}\left[1 - e^{-\frac{\rho Q R^2}{I'}t}\right]}$$