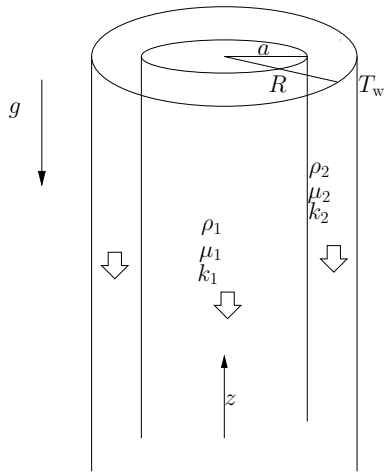
INTRODUCTION TO FLUID MECHANICS. **18-12-2017**
PROBLEM 1 (10 POINTS). **Time: 70 minutes**

NAME:..... **GRADE:.....**

The sketch shows two concentric and parallel flows of different liquids, 1 and 2, along an infinitely long vertical pipe of radius R. Each liquid has a density $\rho_{\rm i}$, a viscosity $\mu_{\rm i}$ and a conductivity $k_{\rm i}$, where ${\rm i} = 1, 2$. If the innner liquid stream has a radius a,

1. Using the continuity equation, show that $v_r = 0$ in both fluids.
2. Show that the only dependence of the pressure field is be of the form $p = p(z)$.
3. In the absence of pressure gradient, find the velocity field $w(r)$ and the volumetric flow rates Q_1 and Q_2 of liquids 1 and 2, respectively.
4. If the wall of the cylinder is maintained at a temperature $T_{\rm w}$, formulate the problem, with all the boundary conditions, that would yield the temperature distribution in the liquids.
5. Use an adequate control volume to determine the heat flux q, per unit area and time, lost to the ambient through the cylinder wall, if this problem makes sense.



① CONTINUITY: $\frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial w}{\partial z} = 0 \Rightarrow r v_r = \text{const}$ ($v_\theta = 0$; NO DEPENDENCE ON z)

→ LIQUID 1: $r=0$: $r v_r = 0 \Rightarrow \boxed{v_r = 0}$

→ LIQUID 2: $r = R$: $v_r = 0 \Rightarrow \boxed{v_r = 0}$

② AXIAL SYMMETRY $\Rightarrow \frac{\partial p}{\partial \theta} = 0$

RADIAL MOMENTUM $\Rightarrow$ $v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_z\frac{\partial v_r}{\partial z} = -\frac{\partial p}{\partial r} + \nu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r v_r)\right) + \cdots\right] + g_r$

($v_r = 0$, $\frac{\partial v_r}{\partial r} = 0$, $\frac{\partial v_r}{\partial \theta} = 0$, $v_\theta = 0$, $\frac{\partial v_r}{\partial z} = 0$, $\frac{\partial}{\partial r}(\frac{1}{r}\frac{\partial}{\partial r}(r v_r)) = 0$, $g_r = 0$)

$\frac{\partial p}{\partial r} = 0$

$\Rightarrow \boxed{p = p(z) \text{ IN BOTH FLUIDS}}$

③ LIQUID 1: $0 = -\frac{\partial p}{\partial z} + \rho g + \frac{\mu_1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right)$; $\frac{\partial w_1}{\partial r} = \frac{\rho_1 g}{\mu_1}\frac{r^2}{2} + \frac{A}{r}$ (A = 0: NO SINGULARITY)

$\boxed{w_1 = \frac{\rho_1 g}{\mu_1}\frac{r^2}{4} + B_1}$

LIQUID 2: SAME EQUATION, $\frac{\partial w_2}{\partial r} = \frac{\rho_2 g}{\mu_2}\frac{r}{2} + \frac{A_2}{r}$

$\boxed{w_2 = \frac{\rho_2 g}{\mu_2}\frac{r^2}{4} + A_2 \ln r + B_2}$

CONSTANTS B_1, A_2, B_2 TO BE DETERMINED WITH B.C:

* AT $r = R$: $w_2 = 0 \Rightarrow 0 = \frac{\rho_2 g R^2}{\mu_2 4} + A_2 \ln R + B_2$

$\boxed{w_2 = \frac{\rho_2 g}{4\mu_2}(r^2 - R^2) + A_2 \ln\frac{r}{R}}$

A_2, B_2 FROM B.C. AT INTERFACE:

* $r = a$: $\rightarrow \bar{n} = \bar{e}_r$ $\left[\left(-p_2\bar{n} + \bar{\bar{\tau}}_2'\cdot\bar{n}\right) - \left(-p_1\bar{n} + \bar{\bar{\tau}}_1'\cdot\bar{n}\right)\right]_{r=a} = 0$ NOTICE

$\bar{\bar{\tau}}_i'\cdot\bar{e}_r = \begin{pmatrix} 0 & 0 & \mu_i\frac{\partial w_i}{\partial r} \\ 0 & 0 & 0 \\ \mu_i\frac{\partial w_i}{\partial r} & 0 & 0 \end{pmatrix}\cdot\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}0\\0\\\mu_i\frac{\partial w_i}{\partial r}\end{pmatrix}$

$-(p_2 - p_1)\bar{e}_r + \left(\mu_2\frac{\partial w_2}{\partial r} - \mu_1\frac{\partial w_1}{\partial r}\right)\bar{e}_z = 0 \Rightarrow \boxed{p_2 = p_1}$

$\boxed{\mu_2\frac{\partial w_2}{\partial r}\Big|_a = \mu_1\frac{\partial w_1}{\partial r}\Big|_a}$ } 2 CONDICIONES

$\boxed{w_2 = w_1}$

$\mu_2\frac{\partial w_2}{\partial r}\Big|_a = \mu_1\frac{\partial w_1}{\partial r}\Big|_a \Rightarrow \mu_1\frac{\rho_1 g a}{2\mu_1} = \mu_2\frac{\rho_2 g a}{2\mu_2} + \mu_2\frac{A_2}{a}$ } $\rightarrow \boxed{A_2 = \frac{(\rho_1 - \rho_2) g a^2}{\mu_2}}$

$w_1|_a = w_2|_a \Rightarrow \frac{\rho_1 g a^2}{4\mu_1} + B_1 = \frac{\rho_2 g}{4\mu_2}(a^2 - R^2) + A_2 \ln\frac{a}{R}$

$\boxed{B_1 = \frac{\rho_2 g a^2}{4\mu_2}\left(1 - \left(\frac{R}{a}\right)^2\right) - \frac{\rho_1 g a^2}{4\mu_1} + \frac{(\rho_1 - \rho_2) g a^2}{\mu_2}\ln\frac{a}{R}}$

$$w_1 = \frac{\rho_1 g (r^2 - a^2)}{4\mu_1} + \frac{\rho_2 g a^2}{4\mu_2}\left(1 - \left(\frac{R}{a}\right)^2\right) + \frac{(\rho_1 - \rho_2) g a^2 \ln \frac{a}{R}}{\mu_2}$$

$$w_2 = -\frac{\rho_2 g}{4\mu_2}(R^2 - r^2) + \frac{(\rho_1 - \rho_2) g a^2 \ln \frac{r}{R}}{\mu_2}$$

$$Q_1 = \int_0^a 2\pi r w_1 \, dr = -\frac{\rho_1 g}{4\mu_1} 2\pi \left[a^2 \frac{r^2}{2} - \frac{r^4}{4}\right]_0^a - \frac{\rho_2 g a^2}{4\mu_2}\left[\left(\frac{R}{a}\right)^2 - 1\right]\pi R^2 + \frac{(\rho_1 - \rho_2) g a^2 \ln \frac{a}{R} \pi R^2}{\mu_2}$$

$$Q_1 = -\frac{\rho_1 g \pi a^4}{8\mu_1} - \frac{\rho_2 g a^2}{4\mu_2}\left[\left(\frac{R}{a}\right)^2 - 1\right]\pi R^2 + \frac{(\rho_1 - \rho_2) g \pi a^2 R^2 \ln \frac{a}{R}}{\mu_2}$$

$$Q_2 = \int_a^R 2\pi r w_2 \, dr = -\frac{\rho_2 g}{4\mu_2} 2\pi \left[R^2 \frac{r^2}{2} - \frac{r^4}{4}\right]_a^R + \frac{(\rho_1 - \rho_2)}{\mu_2} g a^2 \underbrace{\int_a^R \ln \frac{r}{R} \, dr}_{= R\left[x \ln x - x\right]_{\frac{a}{R}}^{1}}$$

$$Q_2 = -\frac{\rho_2 g \pi R^4}{8\mu_2} + \frac{\rho_2 g \pi R^4}{4\mu_2}\left[\left(\frac{a}{R}\right)^2 - \left(\frac{a}{R}\right)^4\right] + \frac{\rho_1 - \rho_2}{\mu_2} g a^2 \left(-R + a - a \ln \frac{a}{R}\right)$$

④ $\Phi_V = \mu \left(\frac{\partial w}{\partial r}\right)^2$; $\Phi_{V_1} = \mu_1 \left(\frac{2\rho_1 g r}{4\mu_1}\right)^2 = \frac{\rho_1^2 g^2 r^2}{4\mu_1}$

ò $\Phi_{V_2} = \mu_2 \left[\frac{2\rho_2 g r}{4\mu_2} + \frac{A_2}{r}\right]^2 = \frac{\rho_2^2 g^2 r^2}{4\mu_2} + \mu_2 \frac{\rho_2 g}{\mu_2} \frac{A_2}{r} + \mu_2 \frac{A_2^2}{r^2}$

ec. energia interna:

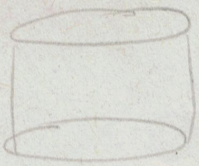
$$\Phi_{V_1} + \frac{k_1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial T_1}{\partial r}\right) = 0 \quad ; \quad r = 0: \frac{\partial T_1}{\partial r} = 0$$

$$\Phi_{V_2} + \frac{k_2}{r}\frac{\partial}{\partial r}\left(r \frac{\partial T_2}{\partial r}\right) = 0 \qquad r = a: T_1 = T_2$$

$$k_1 \frac{\partial T_1}{\partial r}\Big|_a = k_1 \frac{\partial T_2}{\partial r}\Big|_a$$

$$r = R: T = T_w$$

⑤ ec. energia interna entre 2 planos:



$$L\ \frac{d}{dt}\int_{V_C} \rho e\, dV + \int_{\Sigma_C} (\rho e \vec{v} - \vec{r}_t)\cdot \vec{n}\, d\Sigma = \int -\vec{q}\cdot\vec{n}\, d\Sigma + \int_{V_C} (Q_r + Q_{ch})\, dV + \int_{V_C} -P\alpha\, dV + \int_{V_C} \Phi_v\, dV$$

$$-q_w \cdot 2\pi R \cdot L + \int_{V_C} \Phi_v\, dV = 0$$

$$q_w \cdot 2\pi R L = L\int_0^a \Phi_{v_1} 2\pi r\, dr + L\int_a^R \Phi_{v_2} 2\pi r\, dr$$

$$q_w = \frac{1}{R}\int_0^a \Phi_{v_1} r\, dr + \frac{1}{R}\int_a^R \Phi_{v_2} r\, dr$$

$$\int_0^a \Phi_{v_1}\cdot r\, dr = \frac{\rho_1^2 g^2}{\mu_1}\int_0^a r^3 dr = \frac{\rho_1^2 g^2 a^4}{4\mu_1}$$

$$\int_a^R \Phi_{v_2}\, r\, dr = \frac{\rho_2^2 g^2}{\mu_2}\int_a^R r^3 dr + \rho_2 g A_2 \int_a^R r\, dr + \mu_2 A_2^2 \int_a^R \frac{dr}{r}$$

$$= \frac{\rho_2^2 g^2 (R^4 - a^4)}{4\mu_2} + \rho_2 g A_2 \frac{(R^2 - a^2)}{2} + \mu_2 A_2^2 \ln\frac{R}{a}$$

$$\boxed{q_w = \frac{\rho_1^2 g^2 a^4}{4\mu_1 R} + \frac{\rho_2^2 g (R^4 - a^4)}{4\mu_2 R} + \frac{\rho_2 g A_2 (R^2 - a^2)}{2R} + \frac{\mu_2 A_2^2}{R}\ln\frac{R}{a}}$$