INTRODUCTION TO FLUID MECHANICS. **18-12-2017**
PROBLEM 1 (10 POINTS). **Time: 90 minutes**

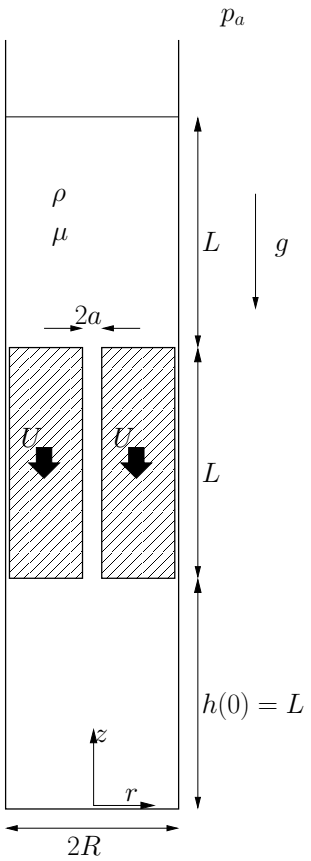
NAME:..... **GRADE:.....**

The sketch shows a piston of radius R and length L inside a cylinder open to the atmosphere, of the same diameter and initially full of a liquid of density ρ and viscosity μ up to a level $3L$. The piston can slide along the cylinder without friction and no leaks exist between the piston and the cylinder walls. The piston has a duct, of diameter $2a \ll R \sim L$, through which the liquid can flow upwards. For $t < 0$, the piston and the liquid are stagnant. If $h(t < 0) = L$,

1. Determine the pressure distribution in the liquid and the net force exerted by the fluid on the piston.

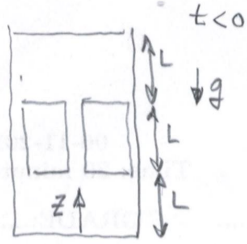
For $t \geq 0$ the piston starts moving in the duct downwards with a constant velocity U. After a short transient, that will be neglected, the flow becomes dominated by viscosity. Using the reference system shown in the sketch to formulate the problem, and while the height $h(t)$ of the piston is of the same order as L,

2. Use the integral continuity equation to estimate the characteristic value for the axial velocity w_c in teh duct.
3. Determine the criteria for the flow inside the duct to be dominated by viscosity
4. Obtain the axial velocity field $w(r, t)$ in the pipe, and the flow rate $Q(t)$ as functions of the reduced pressure gradient $\partial P/\partial z$
5. Determine the reduced pressure distributions in the chambers at the entrance and at the exit of the duct, P_0 and P_1 respectively. Obtain the reduced pressure distribution $P(z)$ along the duct. Neglect the velocities at the bulk of both chambers.
6. Find the force $\bar{F}$ exerted by the fluid on the piston.



Note: In the algebraic expressions, take into account that $a/R << 1$ and therefore a/R is negligible compared with unity.

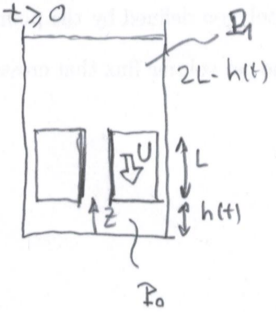
① $t<0$



FLUID STATICS: $-\nabla P + \rho g \hat{k} = 0 \Rightarrow P + \rho g z = \text{const} = P_a + \rho g 3L$

$$\bar{F} = \pi(R^2 - a^2)\cdot L \rho g \hat{k} \simeq \pi R^2 L \rho g \hat{k}$$ BUOYANCY

② $t \geq 0$



$$\frac{O\left(\rho \frac{\partial w}{\partial t}\right)}{O\left(\frac{\mu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right)\right)} \sim \frac{\rho \frac{w_c}{t_c}}{\mu \frac{w_c}{a^2}} \sim \frac{\rho a^2}{\mu t_c} \sim \frac{\rho U a}{\mu}\frac{a}{L} \ll 1 \quad (1)$$

$t_c \sim \frac{L}{U}$ (we expect).

$$\frac{O\left(\rho w \frac{\partial w}{\partial z}\right)}{O\left(\frac{\mu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right)\right)} \sim \frac{\rho \frac{w_c^2}{L}}{\mu \frac{w_c}{a^2}} \sim \frac{\rho w_c a}{\mu}\frac{a}{L} \sim \frac{\rho U R^2}{\mu L} \ll \quad (2)$$

Continuity $\Rightarrow w_c \pi a^2 \sim U \cdot R^2 \pi$

③ DOMINATED BY VISCOSITY $\Rightarrow$

$$\boxed{0 = -\frac{\partial P}{\partial z} + \frac{\mu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right);\ \frac{\partial w}{\partial r}(r=0)=0,\ w(r=a)=-U}$$

$$\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) = \frac{\partial P}{\partial z}\frac{r}{\mu} \rightarrow r\frac{\partial w}{\partial r} = \frac{\partial P}{\partial z}\frac{r^2}{2\mu} + \underbrace{A}_{0} \quad \text{AVOID SINGULARITY}$$

$$\left.\begin{aligned} w &= \frac{\partial P}{\partial z}\cdot\frac{r^2}{4\mu} + B \rightarrow \\ -U &= \frac{\partial P}{\partial z}\cdot\frac{a^2}{4\mu} + B \end{aligned}\right\} \Rightarrow \boxed{w = -U - \frac{1}{4\mu}\frac{\partial P}{\partial z}(a^2 - r^2)}$$

FLOW RATE: $Q = \int_0^a 2\pi r\, w(r)\, dr = -2\pi U\frac{a^2}{2} - \frac{2\pi}{4\mu}\frac{\partial P}{\partial z}\left[a^2\frac{r^2}{2} - \frac{r^4}{4}\right]_0^a$

$$\boxed{Q = -\pi a^2 U + \frac{\pi a^4}{8\mu}\frac{\partial P}{\partial z}} \quad (3)$$

④ (3) CAN BE TURNED INTO A P EQN. $-\frac{\partial P}{\partial z} = \frac{8\mu Q}{\pi a^4} + \frac{8\mu U}{a^2}$, $z \in [h, h+L]$

WHICH CAN BE INTEGRATED: $P_0 - P(z) = \frac{8\mu Q}{\pi a^4}(z-h) + \frac{8\mu U}{a^2}(z-h)$

IN PARTICULAR, AT $z = L+h$: $P_0 - P_1 = \frac{8\mu Q}{\pi a^4}L + \frac{8\mu U}{a^2}L \quad (4)$

IN (4) P_0 IS UNKNOWN, $\boxed{P_1 = \text{const} = P_a + 3\rho g L}$ (5), AND Q CAN BE OBTAINED FROM CONTINUITY



$V_c = \pi R^2 h + \pi a^2 L$; $\frac{d}{dt}\int_{V_c} dV + \int_{\Sigma_c} \rho(\vec{v}-\vec{v}_c)\cdot\hat{n}\, d\Sigma = 0 \rightarrow \pi R^2 \dot{h} + \underbrace{\int_0^a 2\pi r(w+U)\, dr}_{Q} = 0$

$\dot{h} = -U \Rightarrow -\pi R^2 U + Q + \pi a^2 U = 0$

$$\boxed{Q = \pi(R^2 - a^2)U \simeq \pi R^2 U} \quad (6)$$

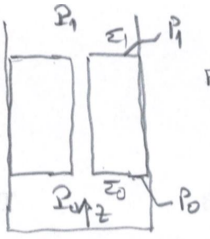
(4),(5),(6) $\Rightarrow$ $P_0 = P_a + 3\rho g L + \frac{8\mu U L}{a^2} + \frac{8\mu L}{\pi a^4}Q$

$$P_0 = P_a + 3\rho g L + \frac{8\mu U L}{a^2} + \frac{8\mu L}{\pi a^4}\cdot\pi R^2 U = P_a + 3\rho g L + \frac{8\mu U L}{a^2}\left[\cancel{1} + \left(\frac{R}{a}\right)^2\right]$$

($\ll$)

$$\boxed{P_0 = P_a + 3\rho g L + \frac{8\mu U L}{a^2}\left(\frac{R}{a}\right)^2}$$

⑤ FORCE : PRESSURE



$$F_P = \int_a^R (P_0 - P_1)\, 2\pi r\, dr \simeq \int_0^R (P_0 - P_1)\, 2\pi r\, dr$$

$$P_0 = P_0 - \rho g h = P_a + 3\rho g L + \frac{8\mu U L}{a^2}\left(\frac{R}{a}\right)^2 - \rho g h$$

$$P_1 = P_1 - \rho g (L+h) = P_a + 3\rho g L - \rho g (L+h)$$

$$F_P = \int_0^R \left[\left(P_a + 3\rho g L + \frac{8\mu U L}{a^2}\left(\frac{R}{a}\right)^2 - \rho g h\right) - \left(P_a + 3\rho g L - \rho g (L+h)\right)\right] 2\pi r\, dr$$

$$= \left(\rho g L \pi R^2 + 8\pi\mu U L \left(\frac{R}{a}\right)^4\right) k$$

VISCOSITY (ALONG CHANNEL)

$$F_V = -\mu \left.\frac{\partial u}{\partial r}\right|_{r=a} \cdot 2\pi a \cdot L = -\frac{1}{2}\frac{\partial P}{\partial z}\, 2\pi a L \Big|_{r=a} = \pi a^2 L \frac{8\mu U}{a^2 L}\left(\frac{R}{a}\right)^2 = 8\pi\mu U L \left(\frac{R}{a}\right)^2 k$$

$$F = \underbrace{\rho g L \pi R^2}_{\text{BUOYANCY}} + \underbrace{8\pi\mu U L \left(\frac{R}{a}\right)^4}_{\text{PRESSURE}} + \underbrace{8\pi\mu U L \left(\frac{R}{a}\right)^2}_{\text{VISCOSITY}} k \simeq \left(\rho g L \pi R^2 + 8\pi\mu U L \left(\frac{R}{a}\right)^4\right) k$$

$\gg$