

a) $\rho\frac{\partial v_z}{\partial t} + \rho v_r\frac{\partial v_z}{\partial r} + \rho v_z\frac{\partial v_z}{\partial z} = -\frac{\partial p}{\partial z} + \frac{1}{r}\mu\frac{\partial}{\partial r}\left(r\cdot\frac{\partial v_z}{\partial r}\right) + \rho g$

$P = p - \rho g z \Rightarrow \quad \rho\frac{\partial v_z}{\partial t} + \rho v_z\frac{\partial v_z}{\partial z} = -\frac{\partial P}{\partial z} + \mu\cdot\frac{1}{r}\frac{\partial}{\partial r}\left(r\cdot\frac{\partial v_z}{\partial r}\right)$

Entry length $\quad \rho\frac{v_{z_c}}{l_e} \sim \mu\frac{v_{z_c}}{R^2} \rightarrow l_e \sim \frac{\rho R^2 v_{z_c}}{\mu}$

$\mu\frac{v_{z_c}}{R^2} \sim \rho g \rightarrow v_{z_c} \sim \frac{\rho g R^2}{\mu}$

$\Big\} \; l_e \sim \frac{\rho R^2}{\mu}\cdot\frac{\rho g R^2}{\mu}$

$l_e \sim \frac{g R^4}{\nu^2} \ll L$

Discharge time : $t_d \sim \frac{V}{v_{z_c}\cdot R^2} \sim \frac{V\nu}{gR^4}$

Unsteady term negligible : $\frac{\rho\cdot v_{z_c}/t_d}{\mu\cdot v_{z_c}/R^2} \sim \frac{\rho R^2}{\mu\frac{V\nu}{gR^4}} \sim \frac{gR^6}{V\nu^2} \ll 1$

b) $0 = -P_l + \frac{\mu}{r}\cdot\frac{\partial}{\partial r}\left(r\cdot\frac{\partial v_z}{\partial r}\right)$

$P_l = \frac{\partial P}{\partial z} = \frac{P(L) - P(0)}{L} = -\rho g\cdot\frac{L+h}{L}$

$P(z=0) = P_a + \rho g h(t)$

$P(z=L) = P_a - \rho g L$

$\Big\} \; v_z(r,z,t) = \frac{-P_l R^2}{4\mu}\left(1 - \left(\frac{r}{R}\right)^2\right)$

$P_l(z) = \frac{\partial P}{\partial z} = -\rho g\cdot\frac{L+h}{L} \Rightarrow P = -\rho g\frac{L+h}{L}z + C$. Using BC : $C = P_a + \rho g h$

$P = P_a + \rho g h - \rho g\cdot\left(1+\frac{h}{L}\right)\cdot z$

c) $\rho\cdot\frac{dV}{dt} = -Q = -2\pi\int_0^R \rho r\cdot v_z\cdot dr = 2\pi\frac{P_l\cdot R^4}{4\mu}\cdot\int_0^1 \rho\,\bar{r}\cdot\left(1-\bar{r}^2\right)\cdot d\bar{r} = \frac{\pi g\cdot P_l\cdot R^4}{8\mu}$

$A = \frac{V}{H} \qquad \frac{1}{\pi}\frac{V}{H}\frac{dh}{dt} = -\rho g\frac{L+h}{L}\frac{R^4}{8\mu} = -\rho g\left(1+\frac{h}{L}\right)\cdot\frac{R^4}{8\mu}$

$\frac{1}{\pi}\frac{d(h/H)}{1+\frac{H}{L}\frac{h}{H}} = -\frac{\rho g R^4}{8V\mu} \rightarrow \quad \ln\left(1+\frac{H}{L}\cdot\frac{h}{H}\right)\frac{L}{H} = -\frac{\pi\rho g R^4 t}{8\mu\cdot V} + \frac{L}{H}\ln\left(1+\frac{H}{L}\right)$

$t_d = \frac{8\cdot\nu\cdot V}{\pi g R^4}\frac{L}{H}\cdot\ln\left(1+\frac{H}{L}\right)$

1) $D = f(\mu, R, U_\infty, \rho)$

$[D] = M L T^{-2}$

$[\mu] = M L^{-1} T^{-1}$

$[U_\infty] = L T^{-1}$

$[\rho] = M L^{-3}$

$[R] = L$

Flow dominated by viscosity $\Rightarrow$

$[D] = [\mu]^\alpha [U_\infty]^\beta [R]^\gamma$

$M L T^{-2} = (M L^{-1} T^{-1})^\alpha (L T^{-1})^\beta \cdot L^\gamma$

M: $1 = \alpha$ $\quad \gamma = 1$

L: $1 = -\alpha + \beta + \gamma$ $\quad \beta = 1$

T: $-2 = -\alpha - \beta$

$$\frac{D}{\mu \cdot R \cdot U_\infty} = f\left(\frac{\rho U_\infty R}{\mu}\right) \quad (1)$$

2) $\rho \cdot \vec{v}' \cdot \nabla \vec{v}' = -\nabla p' + \mu \cdot \nabla^2 \vec{v}'$

$\rho \cdot \frac{U_\infty^2}{R} \quad \frac{\Delta p'}{R} \quad \mu \cdot \frac{U_\infty}{R^2}$

$\begin{cases} \bar{x}' = \bar{x}'_s \rightarrow \bar{v} = 0 \\ \bar{x}' \rightarrow \infty \rightarrow \bar{v} = U_\infty \cdot \bar{e}_x \end{cases}$

Defining:

$\bar{v} = \frac{\bar{v}'}{U_\infty}$

$\bar{x} = \frac{\bar{x}'}{R}$

$\Delta p = \frac{\Delta p'}{\mu \cdot U_\infty / R}$

$Re \cdot \bar{v} \cdot \nabla \bar{v} = -\nabla p + \nabla^2 \bar{v}$ $\quad \begin{cases} \bar{x} = 1 \rightarrow \bar{v} = 0 \\ \bar{x} \rightarrow \infty \rightarrow \bar{v} = 1 \cdot \bar{e}_x \end{cases}$

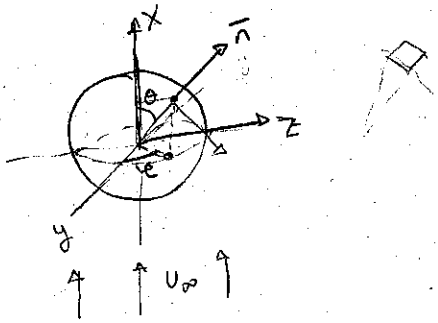
In the limit $Re \rightarrow 0$ $\Rightarrow$ Equations become independent of Reynolds

$0 = -\nabla p + \nabla^2 \bar{v}$

$\Rightarrow$ From (1)

3) In the limit $Re \rightarrow 0$ equations are independente of Reynold $\Rightarrow$ From (1)

$$\frac{D}{\mu R U_\infty} = \Lambda = cte$$



4) $\bar{F} = -\int_{\Sigma_s} (p - p_\infty)\, \bar{n}\, d\sigma + \int_{\Sigma_s} \bar{\bar{\tau}} \cdot \bar{n}\, d\sigma$

$D = F_x = -\int (p - p_\infty) \cdot \cos\theta \, d\sigma + \int_\Sigma (\tau_{rr} \cos\theta - \tau_{r\theta} \operatorname{sen}\theta)\, d\sigma$

$d\sigma = R d\theta \cdot R \operatorname{sen}\theta \cdot d\varphi$

$\tau_{rr} = 0$

$\tau_{r\theta} = -\frac{3\mu}{2R} \cdot U_\infty \cdot \operatorname{sen}\theta$

$p - p_\infty = -\frac{3\mu}{2R} U_\infty \cdot \cos\theta$

$D = \int_{\Sigma_c} \frac{3\mu \cdot U_\infty}{2R} \cos^2\theta \, d\sigma + \int_{\Sigma_c} \frac{3\mu U_\infty^2}{2R} \operatorname{sen}^2\theta \, d\sigma$

$D = \frac{3\mu \cdot U_\infty}{2R} \int_{\Sigma_c} d\sigma = \frac{3\mu \cdot U_\infty R^2}{2R} \int_0^\pi \int_0^{2\pi} \operatorname{sen}\theta \, d\theta \, d\varphi = 6 \cdot \pi \cdot \mu \cdot U_\infty R$

$$\frac{D}{\mu \cdot U_\infty R} = 6 \cdot \pi = \Lambda$$