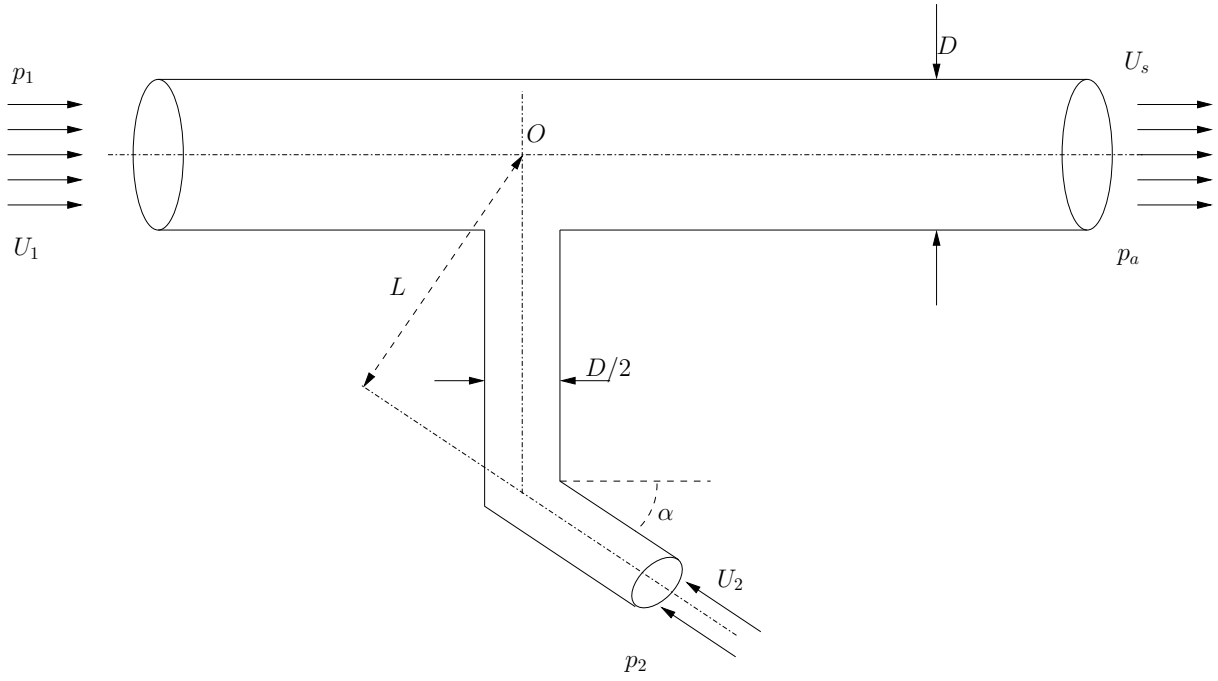
ESCUELA POLITÉCNICA SUPERIOR

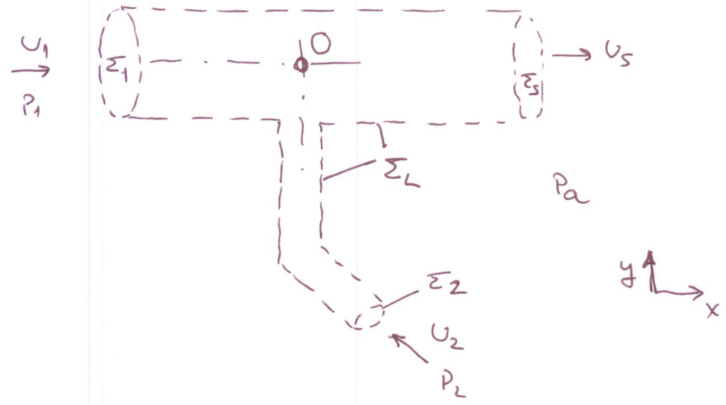
INTRODUCTION TO FLUID MECHANICS. **16-12-2015**
PROBLEM 1 (10 POINTS). **Time: 60 minutes**

NAME:..... **GROUP:.....**

A liquid of density ρ and viscosity μ enters the pipe junction sketched in the figure below. At the entrance sections 1 and 2, with diameters D and $D/2$ respectively, the pressures are p_1 and p_2 and the velocities are U_1 and U_2, respectively. The pipe discharges to the atmosphere (pressure p_a) through section s. It is assumed that the flow is uniform at entries and exits.

1. Determine the value of the velocity at the exit section of the pipe, U_s.
2. Obtain the net force on the duct $\vec{F} = (F_x, F_y)$, as a function of the angle α.
3. Determine the net moment (about point O in the figure) that the fluid exerts on the junction. Write the result as a function of α.





* CONTINUIDAD:

$$\frac{d}{dt}\int_{V_c}\rho\,dV + \int_{\Sigma_c}\rho(\bar{v}-\bar{v}_c)\cdot\bar{n}\,d\sigma = 0$$

0 estacionario

$$\left.\begin{aligned} \Sigma_1 &: -\rho U_1 \pi \frac{D^2}{4} \\ \Sigma_2 &: -\rho U_2 \pi \frac{D^2}{16} \\ \Sigma_3 &: \rho U_3^2 \pi \frac{D^2}{4} \\ \Sigma_L &: 0 \end{aligned}\right\} \Rightarrow \boxed{U_3 = U_1 + \frac{U_2}{4}}$$

* CANTIDAD DE MOVIMIENTO: Usamos manométricas $\Rightarrow$ incluimos efecto atmósfera.

$$\frac{d}{dt}\int_{V_c}\rho\bar{v}\,dV + \int_{\Sigma_1+\Sigma_2+\Sigma_3+\Sigma_L}\rho\bar{v}(\bar{v}-\bar{v}_c)\cdot\bar{n}\,d\sigma = \int_{\Sigma_1+\Sigma_2+\Sigma_3+\Sigma_L}-(P-P_a)\bar{n}\,d\sigma + \int_{\Sigma_1+\Sigma_2+\Sigma_3+\Sigma_L}\bar{\bar{\tau}}'\cdot\bar{n}\,d\sigma + \int_{V_c}\rho\bar{f}_m\,dV$$

0 estacionario; $\Sigma_1+\Sigma_2+\Sigma_3$: despreciables; Σ_L: $-\bar{F}$; $\bar{f}_m$: consideramos despreciable.

$$-\rho U_1^2\pi\frac{D^2}{4}\bar{e}_x + \rho U_3^2\pi\frac{D^2}{4}\bar{e}_x - \rho U_2^2\pi\frac{D^2}{16}(-\cos\alpha\,\bar{e}_x + \operatorname{sen}\alpha\,\bar{e}_y) = (P_1-P_a)\pi\frac{D^2}{4}\bar{e}_x + (P_2-P_a)\pi\frac{D^2}{16}(-\cos\alpha\,\bar{e}_x + \operatorname{sen}\alpha\,\bar{e}_y) - \bar{F}$$

$$\bar{F} = (F_x, F_y) = \frac{\pi D^2}{4}\left\{\left[\rho U_1^2 - \rho U_3^2 - \rho U_2^2\cos\alpha + (P_1-P_a) - \frac{(P_2-P_a)}{4}\cos\alpha\right]\bar{e}_x + \left[\frac{\rho U_2^2}{4}\operatorname{sen}\alpha + \frac{(P_2-P_a)}{4}\operatorname{sen}\alpha\right]\bar{e}_y\right\}$$

* MOMENTO CINÉTICO RESPECTO A O:

$$\frac{d}{dt}\int_{V_c}(\bar{x}-\bar{x}_0)\wedge\rho\bar{v}\,dV + \int_{\Sigma_1+\Sigma_2+\Sigma_3+\Sigma_L}(\bar{x}-\bar{x}_0)\wedge\rho\bar{v}(\bar{v}-\bar{v}_c)\cdot\bar{n}\,d\sigma = \int_{\Sigma_1+\Sigma_2+\Sigma_3+\Sigma_L}(\bar{x}-\bar{x}_0)\wedge(-(P-P_a)\bar{n})\,d\sigma + \int_{\Sigma_1+\Sigma_2+\Sigma_3+\Sigma_L}(\bar{x}-\bar{x}_0)\wedge(\bar{\bar{\tau}}'\cdot\bar{n})\,d\sigma + \int_{V_c}(\bar{x}-\bar{x}_0)\wedge\rho\bar{f}_m\,dV$$

0 estacionario; $\Sigma_1+\Sigma_2+\Sigma_3$: despreciables; Σ_L: $-\bar{M}_0$; $\bar{f}_m$: despreciable.

$$\rho U_2^2\frac{\pi D^2}{16}L\,\bar{e}_z = -(P_2-P_a)\frac{\pi D^2}{16}L\,\bar{e}_z - \bar{M}_0$$

$$\bar{M}_0 = -\left[(P_2-P_a) + \rho U_2^2\right]\frac{\pi D^2}{16}L\,\bar{e}_z$$

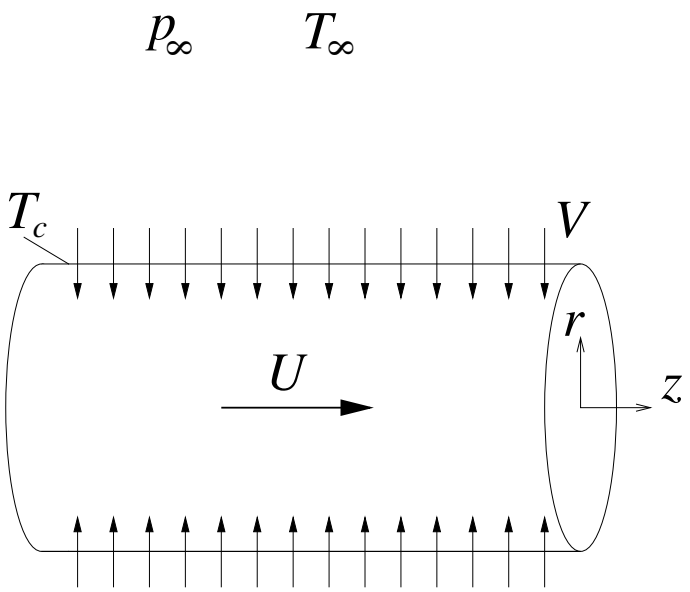
ESCUELA POLITÉCNICA SUPERIOR

INTRODUCTION TO FLUID MECHANICS. **16-12-2015**
PROBLEM 2 (10 POINTS). **Time: 60 minutes**

NAME:..... **GROUP:.....**

An infinitely long porous cylinder of radius R is surrounded by a liquid of density ρ, viscosity μ, specific heat c, and thermal conductivity k. The cylinder moves axially with velocity $v_z = U$ and entrains liquid through its surface pores with a suction velocity $v_r = -V$, inducing for $r > R$ an axisymmetric velocity field with components $v_r(r)$, $v_z(r)$ and $v_\theta = 0$. At large radial distances $r \gg R$ from the cylinder the liquid is at rest with uniform pressure p_∞. Assume in the following analysis that the effect of gravity on the fluid motion is negligible.

1. Use the continuity equation to determine the radial velocity component $v_r(r)$ for $R \leq r < \infty$.
2. From the radial component of the momentum equation compute the pressure distribution, demonstrating in particular that $\partial p/\partial z = 0$.
3. Determine the axial velocity distribution $v_z(r)$ as a function of the suction Reynolds number $Re = \rho V R/\mu$ (to solve the problem, you may look for independent solutions of the resulting linear ODE of the form $v_z \propto r^\alpha$, obtaining the two values of α that are compatible with the equation).
4. Obtain the force acting on the cylinder (per unit length).
5. Calculate the trajectory of the fluid particle located initially at $r = r_o$ and $z = 0$.
6. Compute the temperature field $T(r)$ when the cylinder temperature T_c is different from that of the surrounding atmosphere $T_\infty > T_c$. In the computation, neglect viscous dissipation and express the result with use made of the Prandtl number $Pr = \mu c/k$.
7. Determine the heat transfered to the cylinder per unit time (per unit length).

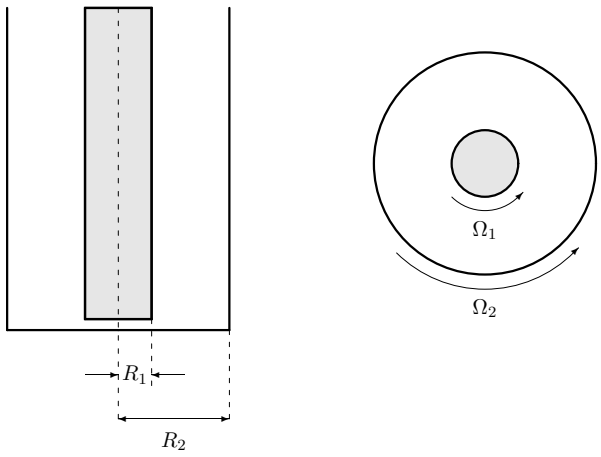


ESCUELA POLITÉCNICA SUPERIOR

INTRODUCTION TO FLUID MECHANICS. **16-12-2015**
PROBLEM 3 (10 POINTS). **Time: 60 minutes**

NAME:..... **GROUP:.....**

The viscosimeter shown in the figure is made of two cylinders of radii R_1 and R_2 rotating at angular velocities Ω_1 and Ω_2, respectively. We need to determine the torque per unit length, M, necessary to sustain the rotating motion of the inner cylinder. Between both cylinders there is an incompressible fluid of density ρ and viscosity μ.



1. Use the Π theorem, reduce the parametric dependence of the torque per unit length M
2. Simplify the previous result for the case $R_2 \gg R_1$ and $\Omega_2 = 0$.

The results of a series of tests, carried out for a cylinder of radious $R_1 = 1$ cm using glycerine as working fluid, $\mu = 1435 \cdot 10^{-3}$, $\rho = 1259$ (units given in SI), are summarized in the following table:

M (N·m/m)	Ω_1 (rpm)
$1.8 \cdot 10^{-4}$	6
$3.6 \cdot 10^{-4}$	12
$9 \cdot 10^{-4}$	30
$18 \cdot 10^{-4}$	60

3. Find the torque needed if, using glycerine, we fix the angular velocity at Ω_1= 6 rpm for the following rotor radii R_1= 1 cm, 1.41 cm, 2.24 cm y 3.16 cm.
4. Using the results in the above table, find the torque needed to rotate a cylinder of radius $R_1 = 1$ cm at angular velocities Ω_1= 1.86, 3.72, 9.30 y 18.6 rpm, if we use SAE 30 oil as working fluid ($\mu = 309 \cdot 10^{-3}$, $\rho = 875$, with units given in SI)

SOLUTION

1. $M = M(\rho, \mu, \Omega_1, \Omega_2, R_1, R_2) \quad \rightarrow \quad \underbrace{\frac{M}{\mu\Omega_1 R_1^2}}_{\Pi_M} = f\Big(\underbrace{\frac{\rho\Omega_1 R_1^2}{\mu}}_{\Pi_\mu}, \underbrace{\frac{\Omega_2}{\Omega_2}}_{\Pi_{\Omega_1}}, \underbrace{\frac{R_2}{R_1}}_{\Pi_{R_2}}\Big)$

   $\Pi_M \equiv$ Torque parameter, $\quad \Pi_\mu \equiv Re \equiv$ Reynolds number,
   $\Pi_{\Omega_2} \equiv$ Angular velocities ratio, $\quad \Pi_{R_2} \equiv$ Radii ratio.

   We chose as dimensionally independent magnitudes μ, Ω_1 y R_1. It **must** be checked tha they are actually dimensionaly independent. The choice of μ is based on the fact that in order to measure viscosity, viscosity has to be necessarily a fundamental parameter of the problem. Ω_1 y R_1 are chosen due to its relevance in the flow and geometry definition (if we read the whole problem we can realize that Ω_2 y R_2 are not so relevant)

2. $\Omega_2 = 0 \Rightarrow \dfrac{\Omega_2}{\Omega_1} = 0$; $R_2 \rightarrow \infty \Rightarrow \dfrac{R_2}{R_1} \rightarrow \infty \quad \Rightarrow \quad \Pi_M = f\Big(Re, 0, \infty\Big) = g(Re)$

   In this case, parametric dependence is reduced to a single parameter.

3. Physical similarity between working conditions a y b, guarantees that if $Re_a = Re_b \Rightarrow \Pi_{M_a} = \Pi_{M_b}$. It can be checked that between corresponding points of the following table we have:
   $\Big(\dfrac{\rho\Omega_1 R_1^2}{\mu}\Big)_a = \Big(\dfrac{\rho\Omega_1 R_1^2}{\mu}\Big)_b \Rightarrow R_{1_b} = R_{1_a}\Big(\dfrac{\Omega_{1_a}}{\Omega_{1_b}}\Big)^{1/2}$,
   and, therefore: $\Big(\dfrac{M}{\mu\Omega_1 R_1^2}\Big)_a = \Big(\dfrac{M}{\mu\Omega_1 R_1^2}\Big)_b \Rightarrow M_b = M_a \dfrac{\Omega_{1_b}}{\Omega_{1_a}}\Big(\dfrac{R_{1_b}}{R_{1_a}}\Big)^2$
   Hence, the table for the new operating conditions becomes:

M_{1_a} (N·m/m)	Ω_{1_a} (rpm)	R_{1_a} (cm)	M_{1_b} (N·m/m)	Ω_{1_b} (rpm)	R_{1_b} (cm)
$1.8 \cdot 10^{-4}$	6	1	$1.8 \cdot 10^{-4}$	6	1
$3.6 \cdot 10^{-4}$	12	1	$3.6 \cdot 10^{-4}$	6	1.41
$9 \cdot 10^{-4}$	30	1	$9 \cdot 10^{-4}$	6	2.24
$18 \cdot 10^{-4}$	60	1	$18 \cdot 10^{-4}$	6	3.16

4. In this case, the similarity conditions are the same as above, but we must take into account that the working fluid is different.
   $\Big(\dfrac{\rho\Omega_1 R_1^2}{\mu}\Big)_{glicerina} = \Big(\dfrac{\rho\Omega_1 R_1^2}{\mu}\Big)_{SAE} \Rightarrow \Big(\dfrac{M}{\mu\Omega_1 R_1^2}\Big)_{glicerina} = \Big(\dfrac{M}{\mu\Omega_1 R_1^2}\Big)_{SAE}$. Since the radii are the same in both cases, we can check that between corresponding points we have: $\dfrac{\Omega_{1_g}}{\nu_g} = \dfrac{\Omega_{1_{SAE}}}{\nu_{SAE}}$ and, therefore, $M_{SAE} = M_g \frac{\mu_{SAE}}{\mu_g} \frac{\Omega_{1_{SAE}}}{\Omega_{1_g}} \Big(\frac{R_{1_{SAE}}}{R_{1_g}}\Big)^2$.
   The table with the new operating conditions is now:

M_{1_g} (N·m/m)	Ω_{1_g} (rpm)	R_{1_g} (cm)	$M_{1_{SAE}}$ (N·m/m)	$\Omega_{1_{SAE}}$ (rpm)	$R_{1_{SAE}}$ (cm)
$1.8 \cdot 10^{-4}$	6	1	$0.12 \cdot 10^{-4}$	1.86	1
$3.6 \cdot 10^{-4}$	12	1	$0.24 \cdot 10^{-4}$	3.72	1
$9 \cdot 10^{-4}$	30	1	$0.60 \cdot 10^{-4}$	9.30	1
$18 \cdot 10^{-4}$	60	1	$1.20 \cdot 10^{-4}$	18.6	1

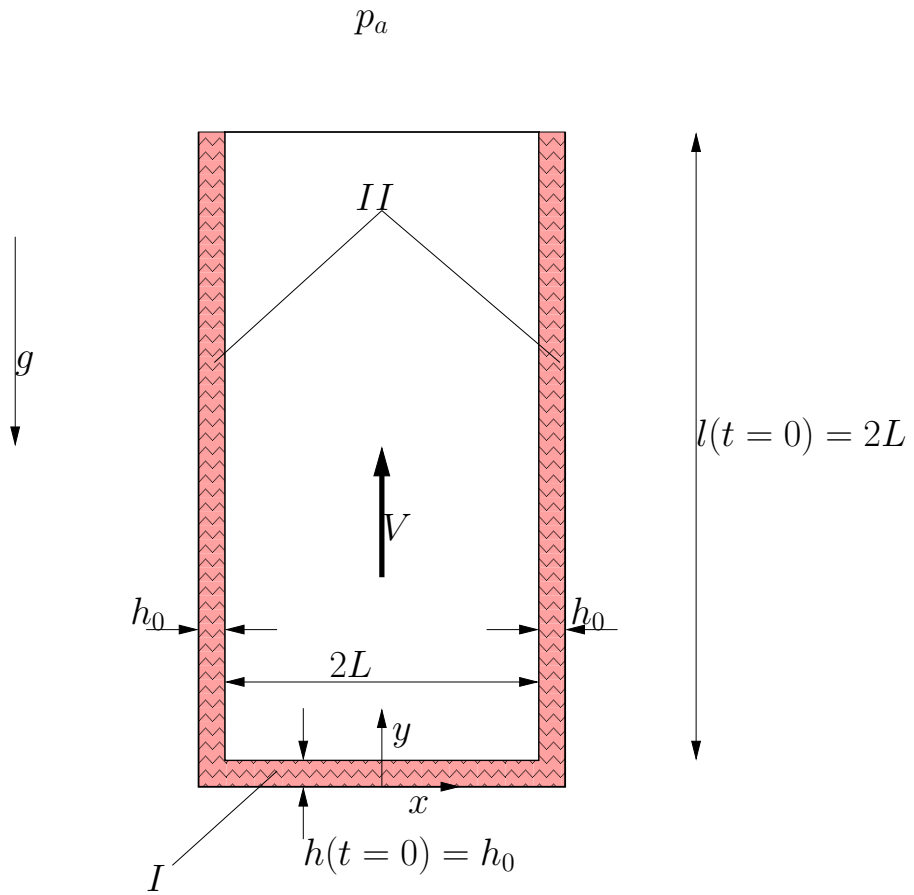
ESCUELA POLITÉCNICA SUPERIOR

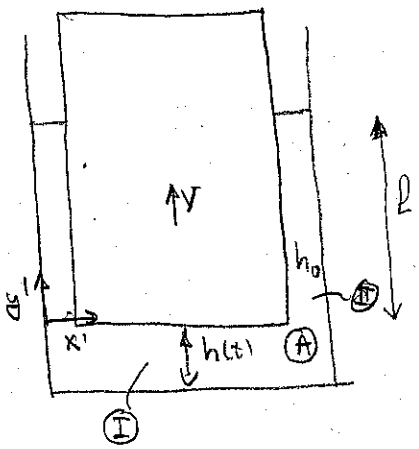
INTRODUCTION TO FLUID MECHANICS. **16-12-2015**
PROBLEM 4 (10 POINTS). **Time: 60 minutes**

NAME:..... **GROUP:.....**

We want to lift the 2-dimensional body shown, in the initial position, in the figure. A layer of oil exists between the body and the container solid surface. The inflow caused by the lifting motion of the body is associated with the existence of underpressures in the liquid film, inducing an adhesion force to be computed here for the case of constant velocity $V = dh/dt$. Taking into account that two slender regions, I and II, are generated, because in the time interval of interest $h_0 \leq h(t) \ll L$,

- Using the mass conservation principle, find $l(t)$.
- Give the criteria that must be satisfied for the flow to be dominated by viscosity in regions I and II. Use only data of the problem.
- Obtain the x component of the velocity in region I and the y component of the velocity in region II, as functions of $\partial P/\partial x$ and $\partial P/\partial y$, respectively.
- Using continuity, derive equations for $P(x,t)$ in region I and for $P(y,t)$ in region II.
- Compute the pressure distribution as well as the **total** vertical force acting on the body.





① CONTINUITY :

VOLUME = VOLUME$_0$

$$2(L+h_0)h_0 + 2l_0h_0 = 2(L+h_0)h + 2lh_0$$

$$l = l_0 + \frac{L}{h_0}(h_0-h) + (h_0-h) \simeq l_0 + \frac{L}{h_0}(h_0-h)$$

$$l(t) \simeq l_0 + \frac{L}{h_0}(h_0 - Vt)$$

$$\frac{dl}{dt} = -\frac{L}{h_0}V \gg V$$

---

② (I)

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\frac{u_c}{L} \sim \frac{v_c}{h_0} \; ; \; v_c \sim V \rightarrow u_c \sim V\frac{L}{h_0}$$

$$\frac{\rho \frac{\partial u}{\partial t}}{\mu \frac{\partial^2 u}{\partial y^2}} \sim \frac{\rho \frac{u_c}{t_0}}{\mu \frac{u_c}{h_0^2}} = \frac{\rho h_0^2}{\mu t_0} = \frac{\rho h_0^2}{\mu \frac{h_0}{V}} = \frac{\rho V h_0}{\mu} \ll 1$$

$$t_0 \sim \frac{h_0}{V}$$

$$\frac{\rho \vec{v}\cdot\nabla\vec{v}}{\mu \frac{\partial^2 u}{\partial y^2}} \sim \frac{\rho \frac{u_c^2}{L}}{\mu \frac{u_c}{h_0^2}} = \frac{\rho u_c h_0^2}{\mu L} = \frac{\rho V h_0}{\mu} \ll 1$$

(II)

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad v_c \sim \frac{dl}{dt} = \frac{L}{h_0}V$$

$$\frac{u_c}{h_0} \sim \frac{v_c}{L} \qquad u_c \sim v_c\frac{h_0}{L} \sim V$$

$$\frac{\rho \frac{\partial v}{\partial t}}{\mu \frac{\partial^2 v}{\partial x^2}} \sim \frac{\rho \frac{v_c}{t_0}}{\mu \frac{v_c}{h_0^2}} = \frac{\rho h_0^2}{\mu t_0} \sim \frac{\rho h_0 V}{\mu} \ll 1$$

$$t_0 \sim \frac{L}{v_c} \sim \frac{h_0}{V}$$

$$\frac{\rho \vec{v}\cdot\nabla\vec{v}}{\mu \frac{\partial^2 v}{\partial x^2}} \sim \frac{\rho \frac{v_c^2}{L}}{\mu \frac{v_c}{h_0^2}} = \frac{\rho v_c h_0^2}{\mu L} = \frac{\rho V h_0}{\mu} \ll 1.$$

③

$$0 = -\frac{\partial P}{\partial x} + \mu\frac{\partial^2 u}{\partial y^2}$$

$y=0 : u=0$

$y=h(t) : u=0$

$$\boxed{u = -\frac{1}{2\mu}\frac{\partial P}{\partial x}y(h-y)}$$

$$0 = -\frac{\partial P}{\partial y'} + \mu\frac{\partial^2 v}{\partial x^2}$$

$x'=0 : v=0$

$x'=h_0 : v=V$

$$v = \frac{1}{2\mu}\frac{\partial P}{\partial y'}x'^2 + Ax' + \cancel{B}^{\,0}$$

$$x'=h_0 : V = \frac{1}{2\mu}\frac{\partial P}{\partial y'}h_0^2 + Ah_0$$

$$\boxed{v = -\frac{1}{2\mu}\frac{\partial P}{\partial y'}x'(h_0-x') + \frac{V}{h_0}x'}$$

④

$$\boxed{Q_I = \int_0^{h(t)} u\,dy = -\frac{1}{2\mu}\frac{\partial P}{\partial x}\left[h\frac{y^2}{2} - \frac{y^3}{3}\right]_0^h = \frac{-h^3}{12\mu}\frac{\partial P}{\partial x}}$$

$$Q_{II} = \int_0^{h_0} v\,dx' = -\frac{1}{2\mu}\frac{\partial P}{\partial y}\left[h_0\frac{x'^2}{2} - \frac{x'^3}{3}\right]_0^{h_0} + \frac{V}{2h_0}x'^2\Big|_0^{h_0}$$

$$\boxed{Q_{II} = -\frac{h_0^3}{12\mu}\cdot\frac{\partial P}{\partial y} + \frac{Vh_0}{2}}$$

$Q_I \neq$ const. ; Continuity :

$$\int_0^h \frac{\partial u}{\partial x}dy + \int_0^h \frac{\partial v}{\partial y}dy = 0 \Rightarrow \frac{\partial Q_I}{\partial x} + V = 0$$

$Q_I(x=0) = 0$

$$Q_I = -Vx$$

$$\frac{\partial P}{\partial x} = -\frac{12\mu Q_I}{h^3} = \frac{12\mu V x}{h^3}$$

$$P_I(L) = P_A \rightarrow P_A - P(x) = \frac{12\mu V}{h^3}x^2\Big|_x^L$$

$$= \frac{12\mu V}{h^3}(L^2 - x^2)$$

$$\boxed{P_I(x) = P_A - \frac{12\mu V}{h^3}(L^2 - x^2)}$$

$0 < x < L$

$Q_{II} = $ const $= h_0\frac{dl}{dt} = -LV$

$$-\frac{\partial P}{\partial y} = \frac{12\mu}{h_0^3}Q_{II} - V\frac{h_0}{2}\frac{12\mu}{h_0^3} = \frac{-6\mu V}{h_0^2}\left(\frac{2L}{h_0} + 1\right)$$

$$-\frac{\partial P}{\partial y} \simeq -\frac{12\mu VL}{h_0^3}$$

$$P_{II}(y=l) = P_a + \rho g l$$

$$P_{II}(y) - (P_a + \rho g l) = +\frac{12\mu VL}{h_0^3}(y - l)$$

$$P_A = P_{II}(y=0) = P_a + \rho g l - \frac{12\mu VL}{h_0^3}l$$

$$\boxed{P_{II}(y) = P_a + \rho g l - \frac{12\mu VL}{h_0^3}(l - y)}$$

$0 < y < l$

PRESSURE FORCE:

$$F_P = 2\int_0^L (P_I - P_a)\, dx\, \bar{e}_y = 2(P_A - P_a)L - 2\cdot\frac{12\mu V}{h^3}\left[L^3 - L^3\frac{2}{3}\right] =$$

$P_I \simeq P_I$

$$F_P = \left(2\rho g \ell L - \frac{24\mu V L^2}{h_0^3}\ell - \frac{24\mu V}{h^3}\frac{L^3}{3}\right)\bar{e}_y$$

$$F_P = \underbrace{2\rho g L \ell} - 24\mu V L^2\left[\frac{\ell}{h_0^3} + \frac{L}{3h^3}\right]\bar{e}_y$$

BUOYANCY.

VISCOUS FORCE ALONG VERTICAL CHANNELS.

DOMINANT: $-\mu\left.\frac{\partial V}{\partial x'}\right|_{x'=h_0} \cdot \ell \cdot 2\,\bar{e}_y$

$$-\mu\left.\frac{\partial V}{\partial x'}\right|_{h_0} = \frac{1}{2}\frac{\partial P_{II}}{\partial y}(h_0 - 2x')\Big|_{x'=h_0} - \frac{\mu V}{h_0} = -\frac{6\mu V L}{h_0^2} - \frac{\mu V}{h_0} \simeq -\frac{6\mu V L}{h_0^2}$$

$\frac{\partial P_{II}}{\partial y} = +\frac{12\mu V}{h_0^2}\frac{L}{h_0}$ $\gg$

$$F_V = -\frac{12\mu V L}{h_0^2}\cdot 2\ell\,\bar{e}_y$$

$$F_Y = 2\rho g L\ell - 24\mu V L^2\left(\frac{\ell}{h_0^3} + \frac{L}{3h^3}\right) - \frac{12\mu V L \ell}{h_0^2}$$

$\gg$

$$\boxed{F_Y = 2\rho g L\ell - 24\mu V L^3\left(\frac{\ell}{L} + \frac{h_0^3}{3h^3}\right)}$$

BUOYANCY. ADHESION FORCE