ESCUELA POLITÉCNICA SUPERIOR

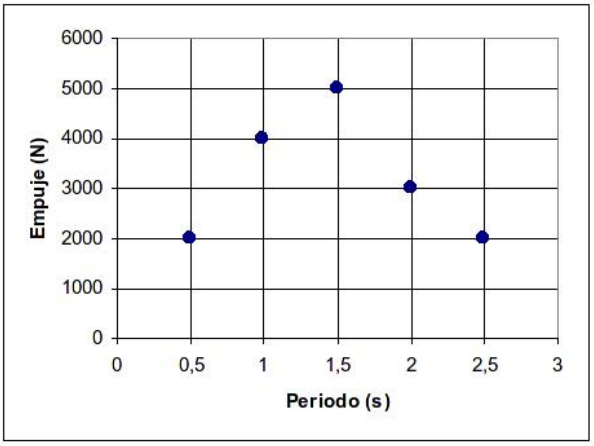
INTRODUCTION TO FLUID MECHANICS. **17-12-2014**
PROBLEM 1 (10 POINTS). **Time: 60 minutes**

NAME:..... **GROUP:.....**

In order to analyze the propulsion of large cetaceans, a detailed study of dolphins is realized in a hydrodynamic channel. In both cases, propulsion is attained by the oscillatory movement of the cetacean's tail, characterized by the amplitude L and the oscillation period T. The oscillation amplitude is directly related to the size of the animal, so the only relevant characteristic length to be considered is L.

1. Use the Π theorem to reduce the parametrical dependence of the thrust produced by the cetaceans. Simplify the result in the case of negligible viscous effects.

A series of experiments is carried out with dolphins (tail movement amplitude $L = 1$ m) in the hydrodynamic channel, at a fixed velocity of the water ($U = 10$ m/s); as a result, the following table of results is obtained:



2. From the results, obtain new data applicable to any cetacean geometrically similar to dolphins and whose propulsion is obtained by oscillatory movements of its tail.

Using previous results:

3. Blue whales usually travel at a velocity $V = 15$ m/s, and their tails oscillate with an amplitude of 3 m and a period of 4 s. What is the thrust generated by the blue whale?

4. If cachalots propel with tail oscillatory movements of amplitude 4 m, what is the oscillation period to generate a thrust of 8000 N if they travel at a velocity of 5 m/s.

5. Orcas move, typically, at a velocity $V = 15$ m/s, with tail amplitude movements of 2 m; Find the maximum thrust and the oscillation period in such conditions.

① $F = F(\rho, \mu, U, L, T)$

	M	L	T
ρ	1	-3	0
μ	1	-1	-1
U	0	1	-1
L	0	1	0
T	0	0	1
F	1	1	-2

$\begin{vmatrix} 1 & -3 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 0 \end{vmatrix} \neq 0$, $\underbrace{\rho, U, L}_{3}$ Dim. Ind.

Π THEOREM : $\Pi_F = f(5-3=2$ PARAM).

$$\frac{F}{\rho U^2 L^2} = f\left(\frac{UT}{L}, \frac{\rho UL}{\mu}\right) \simeq f\left(\frac{UT}{L}\right)$$

$Re \to \infty$
μ effects negligible.

②

DOLPHIN
$L = 1m$, $U = 10 m/s$.

T(s)	F(N)
0.5	2000
1	4000
1.5	5000
2	3000
2.5	2000

$\Rightarrow$

"UNIVERSAL"

$\frac{UT}{L}$	$\frac{F}{\rho U^2 L^2}$
5	2.10^{-2}
10	4.10^{-2}
15	$5\,10^{-2}$
20	$3\,10^{-2}$
25	$2\,10^{-2}$

ASSUME CETACEANS ARE GEOMETRICALLY SIMILAR TO DOLPHINS.

③ $U = 15 m/s$, $L = 3m$, $T = 4s$ $\Rightarrow \frac{UT}{L} = 20 \Rightarrow \frac{F}{\rho U^2 L^2} = 3.10^{-2} \Rightarrow$ **F = 60750 N**

④ $L = 4m$, $U = 5 m/s$, $F = 8000 N$ $\Rightarrow \frac{F}{\rho U^2 L^2} = 2.10^{-2} \Rightarrow \frac{UT}{L} = \begin{cases} 5 \to T = 4s \\ 25 \to T = 20s \end{cases}$ TWO SOLUTIONS!

MIGHT ARGUE CACHALOTS WILL PREFER T = 20s (LESS ENERGETIC COST)

⑤ $L = 2m$, $U = 15 m/s$ $\Rightarrow$ OPTIMUM $\frac{F}{\rho U^2 L^2} = 4.5\,10^{-2}$ at $\frac{UT}{L} = 15$

F = 45000 N at T = 2s

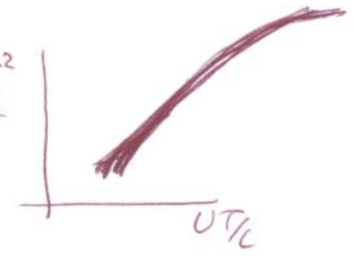
IF DIM. IND. VARIABLES (ρ, L, T):

$$\frac{F}{\rho L^4} T^2 = f\left(\frac{UT}{L}\right)$$

T(s)	F(N)
0.5	2000
1	4000
1.5	5000
2	3000
2.5	2000

$\Rightarrow$

$\frac{UT}{L}$	$\frac{FT^2}{\rho L^4}$
5	0.5
10	4
15	11.25
20	12
25	12.5

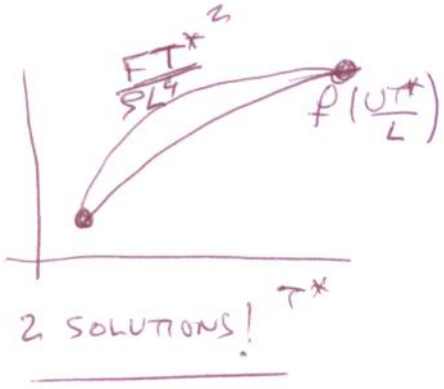


3) $\frac{UT}{L} = 20 \Rightarrow \frac{FT^2}{\rho L^4} = 12 \Rightarrow$ **F = 60750 N**

4) $L = 4\,m$, $F = 8000\,N$, $U = 5\,m/s$ } PROBLEM: FIND T^* SUCH THAT

$$\frac{FT^{*2}}{\rho L^4} = f\left(\frac{UT^*}{L}\right)$$

$\frac{FT^2}{\rho L^4}$	$\frac{UT}{L}$	$T^*(s)$	$\frac{FT^*}{\rho L^4}$
0.5	5	4	0.5
4	10	8	2
14.25	15	12	4.5
12	20	16	8
12.5	25	20	12.5



2 SOLUTIONS!

5) $U_2 = 15\,m/s$, $L_2 = 2\,m$, T_1?, F_2? }

$$\left(\frac{UT}{L}\right)_2 = \left(\frac{UT}{L}\right)_1 \Rightarrow \left(\frac{FT^2}{\rho L^4}\right)_2 = \left(\frac{FT^2}{\rho L^4}\right)_1$$

$$F_2 = F_1 \cdot \left(\frac{L_2}{L_1}\right)^4 \left(\frac{T_1}{T_2}\right)^2, \text{ where } \frac{T_1}{T_2} = \frac{U_2}{U_1}\frac{L_1}{L_2}$$

$$\text{so } F_2 = F_1 \cdot \left(\frac{L_2}{L_1}\right)^2 \left(\frac{U_2}{U_1}\right)^2$$

$$F_{2MAX} = F_{1MAX}\left(\frac{U_2}{U_1}\right)^2\left(\frac{L_2}{L_1}\right)^2 \text{ at } \frac{T_2}{T_1} = \frac{U_1}{U_2}\cdot\frac{L_2}{L_1}$$

$T_2 = 2\,s$

$F_{2max} = 45000\,N$

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INTRODUCTION TO FLUID MECHANICS. **17-12-2014**
PROBLEM 2 (10 POINTS). **Time: 60 minutes**

NAME:..... **GROUP:.....**

A planar body of width $2L$ and mass per unit length M is released at the top of a liquid layer of thickness $h_0 << L$. Because of the gravity, the body descends, displacing the liquid outwards, in the plane $x - y$. If the liquid movement is dominated by viscosity,

1. Using continuity, momentum and boundary conditions, estimate the orders of magnitude of both velocity components (u and v), overpressure ΔP and force exerted by the liquid on the body F_P, as functions of $\dot{h}$, assuming the flow is dominated by viscosity.

2. Using the equation for the vertical movement of the body, estimate the orders of magnitude of terminal velocity $\dot{h}_t$, time to reach terminal velocity t_t, free fall time t_{ff}. In the following three cases, estimate the time for the body to touch the floor, the characteristic velocity $\dot{h}_c$, and give the criterion for the flow to be dominated by viscosity:

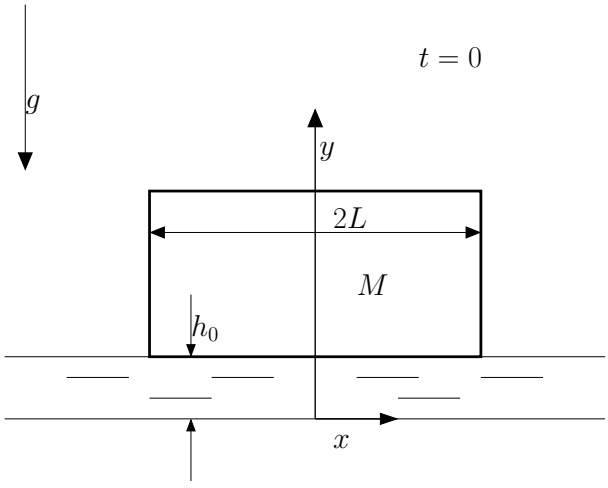
- **a**. $t_t \ll t_{ff}$
- **b**. $t_t \sim t_{ff}$
- **c**. $t_t \gg t_{ff}$

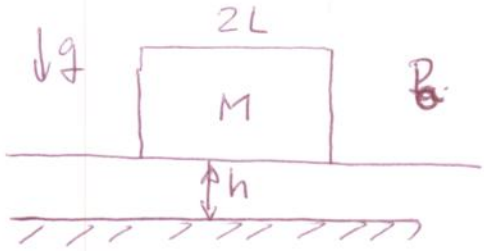
3. Find the velocity distribution as a function of the reduced pressure gradient $\dfrac{\partial P}{\partial x}$.

4. Use the continuity equation to derive an equation for P, and the appropriate boundary conditions. Find P and the force F_P exerted by the fluid on the body.

5. Write the differential equation (with adequate initial conditions) that determines the evolution of h as a function of time. Re-write the equation in non-dimensional form (use h_0 and t_{ff} as scales) in terms of the parameter $\Lambda = \dfrac{8\mu L^3}{M g^{1/2} h_0^{5/2}}$

6. **(extra question worth 2 additional points)** Give a physical interpretation of Λ and find, in the limiting cases $\Lambda \gg 1$ and $\Lambda \ll 1$, the time for the body to reach the floor.





$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$

$$\frac{u_c}{L}\sim\frac{v_c}{h_0}$$

$$\rho\frac{\partial u}{\partial t}+\rho u\frac{\partial u}{\partial x}+\rho v\frac{\partial u}{\partial y}=-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)$$

$$\frac{\rho u_c}{t_0}\qquad\frac{\rho u_c^2}{L}\qquad\qquad\frac{\Delta p}{L}\sim\mu\frac{u_c}{h_0^2}$$

$y=h: v=\dot h=v_c$.

$$F_p=2\int_0^L (p-p_0)\,dx$$

$$v_c\sim\dot h$$

$$u_c\sim\dot h\cdot\frac{L}{h_0}$$

$$\Delta p\sim\mu u_c\frac{L}{h_0^2}\sim\mu\frac{\dot h L^2}{h_0^3}$$

$$F_p\sim\Delta p\cdot L\sim\mu\dot h\frac{L^3}{h_0^3}$$

② $M\frac{d^2h}{dt^2}=-Mg+F_p$

$$M\frac{\dot h}{t_0}\qquad Mg\qquad\mu\dot h\frac{L^3}{h_0^3}$$

$$v_t\sim\frac{Mgh_0^3}{\mu L^3}\;;\quad t_t\sim\frac{v_t}{g}\sim\frac{Mh_0^3}{\mu L^3}$$

$$t_{ff}\sim\left(\frac{h_0}{g}\right)^{1/2}$$

ⓐ $t_t\ll t_{ff}$: $T\sim\frac{h_0}{v_t}\sim\frac{\mu L^3}{Mgh_0^2}$; $\dot h_c\sim\frac{Mgh_0^3}{\mu L^3}=v_t$

ⓑ $t_t\sim t_{ff}$: $T\sim\left(\frac{h_0}{g}\right)^{1/2}$; $\dot h_c\sim(gh_0)^{1/2}$

ⓒ $t_t\gg t_{ff}$: $T\sim\left(\frac{h_0}{g}\right)^{1/2}$; $\dot h_c\sim(gh_0)^{1/2}$

DOMINATED BY VISCOSITY IF:

ⓑ ⓒ $\frac{O\left(\rho\frac{\partial u}{\partial t}\right)}{O\left(\mu\frac{\partial^2 u}{\partial y^2}\right)}\sim\frac{\rho\frac{u_c}{t_0}}{\mu\frac{u_c}{h_0^2}}\sim\frac{\rho h_0^2}{\mu t_0}\sim\frac{\rho h_0^2}{\mu h_0^{1/2}}g^{1/2}=\frac{\rho h_0^{3/2}g^{1/2}}{\mu}\ll 1$

$\frac{O\left(\rho u\frac{\partial u}{\partial x}\right)}{O\left(\mu\frac{\partial^2 u}{\partial y^2}\right)}\sim\frac{\rho u_c h_0^2}{\mu L}\sim\rho\frac{(gh_0)^{1/2}}{\mu}\cdot\frac{K}{K_0}h_0^2\sim\frac{\rho g^{1/2}h_0^{3/2}}{\mu}\ll 1$ } SAME.

ⓐ $\frac{\rho\frac{u_c}{t_0}}{\mu\frac{u_c}{h_0^2}}\sim\frac{\rho h_0^2}{\mu t_0}\sim\frac{\rho h_0^2}{\mu\mu L^3}Mgh_0^2\sim\frac{\rho g M h_0^4}{\mu^2 L^3}\ll 1$

$\frac{\rho u\frac{\partial u}{\partial x}}{\mu\frac{\partial^2 u}{\partial y^2}}\sim\frac{\rho u_c h_0^2}{\mu L}\sim\frac{\rho h_0^2}{\mu L}\cdot\frac{Mgh_0^3}{\mu L^3}\cdot\frac{L}{h_0}\sim\frac{\rho g M h_0^4}{\mu^2 L^3}\ll 1$ } SAME.

③ $0 = -\frac{\partial P}{\partial x} + \mu \frac{\partial^2 u}{\partial y^2}$

$y=0: u=0$

$y=h: u=0$

$$\boxed{u = -\frac{1}{2\mu}\frac{\partial P}{\partial x} y(h-y)}$$

④ $\int_0^h \frac{\partial u}{\partial x} dy + \int_0^h \frac{\partial v}{\partial y} dy = 0 \rightarrow \frac{\partial Q}{\partial x} + \dot{h} = 0, \quad Q = \int_0^h u\, dy.$

$Q = -\frac{h^3}{12\mu} \cdot \frac{\partial P}{\partial x}$; $Q + \dot{h}x = \text{const.} = 0$ $\quad P(x=L) = P_0$; $\frac{\partial P}{\partial x}(x=0) = 0$

$$\frac{\partial P}{\partial x} = \frac{12\mu \dot{h} x}{h^3}$$

$$P = \frac{12\mu \dot{h} x^2}{2h^3} + C$$

$$P_0 = \frac{12\mu \dot{h} L^2}{2h^3} + C$$

$$\Rightarrow \boxed{P - P_0 = -\frac{6\mu \dot{h}}{h^3}(L^2 - x^2)}$$

$$F_P = 2\int_0^L (P - P_0)\, dx = -\frac{12\mu \dot{h}}{h^3}\int_0^L (L^2 - x^2)\, dx = -\frac{8\mu \dot{h} L^3}{h^3}$$

⑤ $M\frac{d^2h}{dt^2} = -Mg - \frac{8\mu L^3}{h^3}\frac{dh}{dt}$

$h(0) = h_0$

$\dot{h}(0) = 0$

$\eta = \frac{h}{h_0}$

$\tau = \frac{t}{t_{ff}}$

$$\frac{d^2\eta}{d\tau^2} + \frac{\Lambda}{\eta^3}\frac{d\eta}{d\tau} = -1$$

$\eta(0) = 1$

$\frac{d\eta}{d\tau}(0) = 0$

⑥ $$\Lambda = \frac{8\mu L^3}{M h_0^{5/2} g^{1/2}} = 8\,\frac{t_{ff}}{t_t}$$

$\Lambda \ll 1$. ($t_{ff} \ll t_t$, the body won't reach terminal velocity, will fall almost freely):

$$\frac{d^2\eta}{d\tau^2} = -1 \rightarrow \eta = 1 - \frac{\tau^2}{2}$$

$\eta = 0 \Rightarrow \tau = \sqrt{2}$

$$\boxed{t = \sqrt{2}\sqrt{\frac{h_0}{g}}}$$

$\Lambda \gg 1$ ($t_{ff} \gg t_t$, the body will fall with terminal velocity)

$\frac{\Lambda}{\eta^3}\frac{d\eta}{d\tau} = -1$ re-scale τ: $\tau_1 = \frac{\tau}{\Lambda}$

$-\frac{1}{\eta^3}\frac{d\eta}{d\tau_1} = 1$ $\quad \eta = \frac{1}{(1+2\tau_1)^{1/2}} \xrightarrow[\tau_1 \to \infty]{} 0$!!

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INTRODUCTION TO FLUID MECHANICS. **17-12-2014**
PROBLEM 3 (10 POINTS). **Time: 60 minutes**

NAME:..... **GROUP:.....**

The figure below shows a flowmeter. It is made of a L-shaped duct with cross sections A_1 and A_2, respectively, at the entrance and at the exit, and total volume V. A ball of masss M, radious R and heat capacity c_s is in equilibrium in an intermediate diffuser; the equlibrium position depends on the flow rate. If a liquid of density ρ, viscosity μ and heat capacity c is injected at the entrance with a velocity U_1 and pressure p_1 and the exit is open to the atmosphere:

1. Find the velocity and pressure at the exit section.

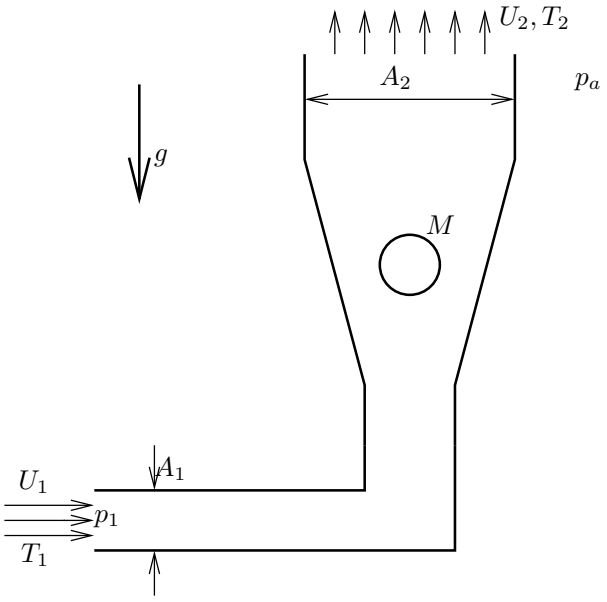
2. Find the force exerted by the **inner liquid** on the walls of the flowmeter.

3. Find the force exerted by the **outer atmosphere** on the walls of the flowmeter.

4. Evaluate the temperature at the exit, T_2, if the walls are adiabatic and the heat transfer per unit time from the ball to the liquid is a constant $\dot{Q}$. Neglect vicous dissipation and chemical and radiative sources of heat.

In order find the temporal evolution of the ball temperature T_B, we decide to include in the analysis the thermal coupling of the solid and the liquid. Take the temperature in the ball to be uniform, $T_B(t) > T_1$ and the heat transfer between the solid and the liquid $\dot{Q} = 4\pi R^2 K(T_B - T_1)$ (K is a constant) and consider the liquid phase flow to be quasi-steady.

5. Write **all** the equations needed to find $T_2(t)$ and $T_B(t)$.



Note: Do not neglect gravity

① 

$$\boxed{P_2 = P_a}$$

$$\cancel{\frac{d}{dt}\int_{V_c}\rho\,dV} + \int_{\Sigma_c}\rho(\bar{v}-\bar{v}_c)\cdot\bar{n}\,d\sigma = 0$$

$$-\rho U_1 A_1 + \rho U_2 A_2 = 0$$

$$\boxed{U_2 = U_1\frac{A_1}{A_2}}$$

② $$\cancel{\frac{d}{dt}\int_{V_c}\rho\,dV} + \int_{\Sigma_c}\rho\bar{v}(\bar{v}-\bar{v}_c)\cdot\bar{n}\,d\sigma = \int_{\Sigma_c}-p\bar{n}\,d\sigma + \int_{\Sigma_c}\bar{\bar{\tau}}'\cdot\bar{n}\,d\sigma + \int_{V_c}\rho\bar{f}\,dV$$

$$-\rho U_1^2 A_1\bar{e}_x + \rho U_2^2 A_2\bar{e}_y = P_1 A_1\bar{e}_x - P_a A_2\bar{e}_y + \rho\left(V-\frac{4}{3}\pi R^3\right)g\,\bar{e}_y$$

$$+\underbrace{\int_{\Sigma_L}-p\bar{n}\,d\sigma + \int_{\Sigma_L}\bar{\bar{\tau}}'\cdot\bar{n}\,d\sigma}_{-\bar{F}_{L\to W}} + \underbrace{\int_{\Sigma_B}-p\bar{n}\,d\sigma + \int_{\Sigma_B}\bar{\bar{\tau}}'\cdot\bar{n}\,d\sigma}_{-\bar{F}_{L\to B}}$$

From equilibrium of the sphere:

$$\bar{F}_{L\to B} - Mg\bar{e}_y = 0 \qquad \bar{F}_{L\to B} = Mg\bar{e}_y$$

$$\boxed{\bar{F}_{L\to W} = (\rho U_1^2 A_1 + P_1 A_1)\bar{e}_x - [\rho U_2^2 A_2 + P_a A_2 + \rho(V-\tfrac{4}{3}\pi R^3)g + Mg\,]\bar{e}_y}$$

③ 

$$\bar{F}_{A\to W} = \int_{\Sigma_L}-P_a\bar{n}\,d\sigma + \int_{\Sigma_1+\Sigma_2}P_a\bar{n} - \underbrace{\int_{\Sigma_1+\Sigma_2}-P_a\bar{n}\,d\sigma}_{0} = \int_{\Sigma_1+\Sigma_2}P_a\bar{n}\,d\sigma =$$

$$\boxed{\bar{F}_{A\to W} = -P_a A_1\bar{e}_x - P_a A_2\bar{e}_y}$$

④ $\frac{d}{dt}\int_{V_c} \rho e \, dV + \int_{\Sigma_c} \rho e (\vec{v} - \vec{v}_c)\cdot \vec{n} \, d\Sigma = \int_{\Sigma_c} -\vec{q}\cdot\vec{n}\, d\Sigma + \int_{V_c} (Q_r + Q) \, dV$

$+ \int_{V_c} \phi_v \, dV + \int_{V_c} p \nabla\cdot\vec{v} \, dV$

$-\rho U_1 A_1 c T_1 + \rho U_2 A_2 c T_2 = \dot{Q}$

$$\boxed{T_2 = T_1 + \frac{\dot{Q}}{\rho U_2 A_2 c T_2}}$$

⑤ Sphere: $\frac{d}{dt}\left[\rho_s \frac{4}{3}\pi R^3 c_s T_B\right] = -\dot{Q}$

$-\rho U_1 A_1 c_1 T_1 + \rho U_2 A_2 c T_2 = \dot{Q}$

$\dot{Q} = 4\pi R^2 K (T_B - T_1)$

$t = 0 : T_B = T_{B0}$

} Unknowns: $T_B, T_2, \dot{Q}$ — 3 eqns.

For the solid:

$$\rho_s \frac{4}{3}\pi R^3 c_s \frac{dT_B}{dt} = -4\pi R^2 K (T_B - T_1)$$

$$\frac{d(T_B - T_1)}{dt} = \frac{-3K}{\rho_s c_s R}(T_B - T_1)$$

$$\ln(T_B - T_1) = -\frac{3Kt}{\rho_s c_s R} + \ln C'$$

$$\boxed{T_B(t) = T_1 + (T_{B0} - T_1) e^{-\frac{3Kt}{\rho_s c_s R}}}$$

$$\boxed{T_2(t) = T_1 + \frac{4\pi R^2 K (T_{B0} - T_1) e^{-\frac{3Kt}{\rho_s c_s R}}}{\rho U_1 A_1 c_1}}$$

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INTRODUCTION TO FLUID MECHANICS. **17-12-2014**
PROBLEM 4 (10 POINTS). **Time: 60 minutes**

NAME:..... **GROUP:.....**

The figure below shows two infinitely long porous cylinders of radii R and $2R$, respectively, aligned with the z axis. The outer cylinder, rotates with an angular velocity $\Omega\vec{e}_z$ while the inner cylinder moves with a velocity $W\vec{e}_z$. A perfect liquid of density ρ, viscosity μ, thermal conductivity k and specific heat c is injected with radial velocity U at the surface of the inner cylinder, filling completely the gap between the cylinders.

1. Find the radial velocity component u. What should be the velocity going out of the domain at the porous outer cylinder for the problem to be well posed?

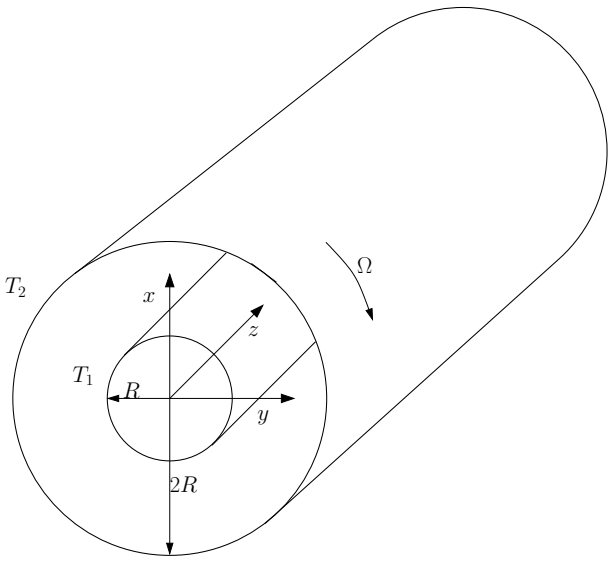
2. Write the equations needed (with boundary conditions) and find the velocity component in the axial direction w, if $\dfrac{\partial p}{\partial z} = 0$

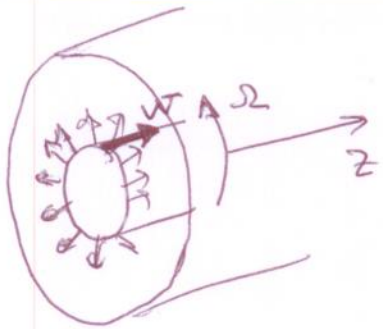
3. Write the equations (with boundary conditions) and find the azimuthal velocity component v.

4. Find the pressure distribution, if the pressure p_1 at the surface of cylinder 1 is known.

5. If the temperatures of cylinders are T_1 and T_2 respectively, find the temperature field T. Neglect viscous dissipation.

6. Find the power needed to maintain the movement of the system.





$(v_r, v_\theta, v_z) = (u, v, w)$

① CONTINUITY:

$$\frac{1}{r}\frac{\partial}{\partial r}(ru) + \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z} = 0$$

($\frac{\partial v}{\partial \theta} = 0$, $\frac{\partial w}{\partial z} = 0$)

$$ru = \text{const.} = RU$$

$$\boxed{u = U\cdot\frac{R}{r}} \rightarrow \text{at } r = 2R: \boxed{u_2 = \frac{U}{2}}$$ To keep mass constant between cylinders

② MOMENTUM Z: $\rho u \frac{\partial w}{\partial r} + \rho w \frac{\partial w}{\partial z} = \frac{\mu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right)$ $\quad r = R: w = W$, $r = 2R: w = 0$

$$\frac{\rho U R}{\mu}\cdot\frac{1}{r}\frac{\partial w}{\partial r} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) \qquad \text{Define } Re = \frac{\rho U R}{\mu}.$$

$$\boxed{\begin{aligned} &Re\frac{dw}{dr} = \frac{d}{dr}\left(r\frac{dw}{dr}\right) \\ &w(R) = W \\ &w(2R) = 0 \end{aligned}} \rightarrow \quad r\frac{dw}{dr} - Re\,w = -A \quad \text{PARTICULAR } w_p = +\frac{A}{Re}$$

HOMOGENEA: $r\frac{dw}{dr} - Re\,w = 0$

$$w_H = B\,r^{Re}$$

$$w = B r^{Re} + \frac{A}{Re} \rightarrow \begin{cases} W = B\cdot R^{Re} + \frac{A}{Re} \\ 0 = B\cdot(2R)^{Re} + \frac{A}{Re} \end{cases} \quad \begin{cases} A = -B\cdot Re\,(2R)^{Re} \\ W = B\left[R^{Re} - (2R)^{Re}\right] \end{cases}$$

$$B = \frac{W}{R^{Re}\left[1 - 2^{Re}\right]} \; ; \; A = -W\,Re\,\frac{R^{Re}\,2^{Re}}{R^{Re}\left[1-2^{Re}\right]} = -W\,Re\,\frac{2^{Re}}{1-2^{Re}}$$

$$\boxed{w = \frac{W}{1-2^{Re}}\left[\left(\frac{r}{R}\right)^{Re} - 2^{Re}\right]}$$

MOMENTUM - θ

$$\rho u \frac{\partial v}{\partial r} + \rho u \frac{v}{r} = \mu \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv)\right).$$

$$\frac{Re}{r}\cdot\left[\frac{dv}{dr} + \frac{v}{r}\right] = \frac{d^2v}{dr^2} + \frac{1}{r}\frac{dv}{dr} - \frac{v}{r^2}$$

$$r^2\frac{d^2v}{dr^2} + r\frac{dv}{dr}\left[1 - Re\right] - \frac{v}{\blacksquare}(1+Re) = 0 \qquad \begin{matrix} v(R) = 0 \\ v(2R) = 2\Omega R \end{matrix}$$

LOOK FOR SOLUTIONS $v = r^n$:

$$n(n-1) + n\cdot(1-Re) - (1+Re) = 0$$

$$n^2 + n(-1 - Re) - (1+Re) = 0$$

$$n = \frac{Re \pm \sqrt{Re^2 + 4(1+Re)}}{2} \quad \begin{matrix} \oplus \rightarrow n_1 = \frac{Re + \sqrt{(Re+2)^2}}{2} = Re+1 \\ \ominus \rightarrow n_2 = \frac{Re - \sqrt{(Re+2)^2}}{2} = -1 \end{matrix}$$

SOLUTION: $v = A\cdot r^{Re+1} + B\,r^{-1}$

$$\left.\begin{matrix} r=0: \; 0 = A\cdot R^{Re+1} + \frac{B}{R^{Re}} \\ r=2R: \; 2\Omega R = A(2R)^{Re+1} + \frac{B}{2R} \end{matrix}\right| \begin{matrix} B = -A\cdot R^{Re+2} \\ 2\Omega R = A\cdot(2R)^{Re+1} - \frac{A}{2R}\cdot R^{Re+2} \end{matrix}$$

$$2\Omega = R^{Re}\cdot\left[2^{Re+1} - \frac{1}{2}\right]A \rightarrow A = \frac{\Omega\cdot 2}{R^{Re}\left(2^{Re+1} - \frac{1}{2}\right)}$$

$$A = \frac{\Omega}{(2R)^{Re}}\,\frac{1}{1 - \frac{1}{2^{Re+2}}} \; ; \; B = -\frac{\Omega R^2}{2^{Re}\left(1 - \frac{1}{2^{Re+2}}\right)}$$

$$v = \frac{4\Omega R}{2^{Re+2} - 1}\cdot\left[\left(\frac{r}{R}\right)^{Re+1} - \frac{R}{r}\right]$$

MOMENTUM - r: $\rho\left(u\frac{\partial u}{\partial r} - \frac{v^2}{r}\right) = -\frac{\partial P}{\partial r}$

$$\frac{d}{dr}\left(P + \tfrac{1}{2}\rho u^2\right) = \rho\frac{v^2}{r} = \frac{4\Omega^2R^2}{(2^{Re+2}-1)^2 R}\,\frac{R}{r}\left[\left(\frac{r}{R}\right)^{2(Re+1)} + \left(\frac{R}{r}\right)^{2} - 2\left(\frac{r}{R}\right)^{Re}\right]$$

$$= \frac{4\Omega^2R^3\rho}{(2^{Re+2}-1)^2}\left[\left(\frac{r}{R}\right)^{2Re+1} + \left(\frac{R}{r}\right)^{3} - 2\left(\frac{r}{R}\right)^{Re-1}\right]$$

$$P + \frac{1}{2}\rho u^2\frac{R^2}{r^2} = \frac{4\Omega^2R^2\rho}{(2^{Re+2}-1)^2}\left[\left(\frac{r}{R}\right)^{2(Re+1)}\cdot\frac{1}{2(Re+1)} + \left(\frac{R}{r}\right)^2\frac{1}{2} - \frac{2}{Re}\left(\frac{r}{R}\right)^{Re}\right] + C$$

at. $r=R$: $P = P_1$:

$$P_1 + \frac{1}{2}\rho U_E^2 = \frac{4\Omega^2R^2\rho}{(2^{Re+2}-1)^2}\underbrace{\left[\frac{1}{2(Re+1)} - \frac{1}{2} - \frac{2}{Re}\right]}_{-\frac{(Re+2)^2}{2Re(Re+1)}} + C$$

$$\boxed{P - P_1 = \frac{1}{2}\rho U^2\left[1 - \left(\frac{R}{r}\right)^2\right] + \frac{4\Omega^2R^2\rho}{(2^{Re+2}-1)^2}\left[\frac{1}{2(Re+1)}\left(\frac{r}{R}\right)^{2(Re+1)} - \frac{1}{2}\left(\frac{R}{r}\right)^2 - \frac{2}{Re}\left(\frac{r}{R}\right)^{Re} + \frac{(Re+2)^2}{2Re(Re+1)}\right]}$$

5/ INTERNAL ENERGY:

$$\rho c u \frac{\partial T}{\partial r} = \frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) \qquad r=R:\ T=T_1 \qquad r=2R:\ T=T_2$$

$$\rho c U R \frac{\partial T}{\partial r} = k\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) \longrightarrow r\frac{dT}{dr} - \underbrace{\frac{\rho c U R}{k}}_{\frac{\rho U R}{\mu c/k}=Re\cdot Pr} = A$$

SOL. PART. $T_p = \frac{A}{Re\,Pr}$

SOL. HOM. $T = B\left(\frac{r}{R}\right)^{Re\,Pr}$

$$\left.\begin{aligned} \frac{r}{R}=1:\ T=T_1 &= \frac{A}{Re\,Pr}+B \\ \frac{r}{R}=2:\ T=T_2 &= \frac{A}{Re\,Pr}+B\,2^{Re\,Pr} \end{aligned}\right\} \quad B=\frac{T_1-T_2}{1-2^{Re\,Pr}} \qquad A=\frac{(T_2-T_1 2^{Re\,Pr})\,Re\,Pr}{1-2^{Re\,Pr}}$$

$$\boxed{T=\frac{1}{1-2^{Re\,Pr}}\cdot\left(T_2-T_1\,2^{Re\,Pr}\right)+\frac{(T_1-T_2)}{1-2^{Re\,Pr}}\left(\frac{r}{R}\right)^{Re\,Pr}}$$

6)



$\bar{F} = \bar{\tau}'\cdot\bar{n}$ $\qquad F_1'' = \mu\frac{\partial w}{\partial r}\bar{e}_z = \mu\frac{W}{R}\frac{1}{1-2^{Re}}\,Re\left(\frac{r}{Re}\right)^{Re-1}\Big|_{r=R}$

$$= \mu\frac{W}{R}\cdot\frac{Re}{1-2^{Re}}$$

$$\boxed{\dot{W}_1 = F_1''\cdot 2\pi R\cdot W = \frac{2\pi\mu Re\,W^2}{1-2^{Re}}}\ ,\ \text{per unit length.}$$

$$F_2'' = \tau_{r\theta} = \mu r\frac{\partial(w/r)}{\partial r} = \mu\cdot r\cdot\frac{4\Omega}{2^{Re+2}-1}\cdot\left[Re\left(\frac{r}{Re}\right)^{Re-1}+2\left(\frac{R}{r}\right)^3\right]$$

$$= \frac{4\mu\Omega}{2^{Re+2}-1}\cdot\left[Re\left(\frac{r}{R}\right)^{Re}+2\left(\frac{R}{r}\right)^2\right] = \frac{4\mu\Omega}{2^{Re+2}-1}\left[Re\,2^{Re}+\frac{1}{2}\right]$$

$$\boxed{\dot{W}_2 = 2\pi(2R)\cdot\Omega\cdot(2R)\cdot\tau_{r\theta}\big|_{2R} = \frac{16\mu\Omega^2R^2}{2^{Re+2}-1}\cdot\left[Re\,2^{Re-1}+1\right]}$$

$\dot{W} = \dot{W}_1+\dot{W}_2$, per unit length.