FLUID MECHANICS I **03-11-2024**
SHORT QUESTIONS 1 (4.5 POINTS) 45%. **Time: 45 minutes**

NAME:..... **GRADE:.....**

EXERCISE 1: Analysis of the flow close to a flat plate

An incompressible fluid flows parallel to a semi-infinite flat plate with a uniform velocity U_0. In this flow, the x component of the velocity can be approximated as:

$$v_x(x,y) = U_0 \left(1 - \exp\left(-\frac{y}{\sqrt{\nu x/U_0}}\right)\right),$$

where ν is the kinematic viscosity of the fluid.

1. Compute the vertical velocity $v_y(x,y)$. **(1 Point)**
2. Compute the tangential component of the viscous forces per unit surface acting on the plate ($y = 0$) at a given x. **(1 Point)**
3. Compute the normal component of the viscous forces per unit surface acting on the plate ($y = 0$) at a given x. **(1 Point)**

Hint:

$$\int \xi \, \exp(-\xi) \, \mathrm{d}\xi = -\xi \, \exp(-\xi) - \exp(-\xi) + C.$$

EXERCISE 2: Pressure close to a tornado's core

A typical tornado has a core radius of about $R = 80$ m at ground level. At the core's boundary, the velocity is around $V_0 = 180$ km/h (50 m/s). We can assume that the fluid motion around the core is perfectly circular with a speed that decays with the distance to the center as r^{-1}:

$$v_\theta(r) = \frac{\Gamma}{2\pi r},$$

where Γ is the tornado's circulation.

1. Compute the circulation Γ using the data provided. **(0.5 Point)**
2. Show that the flow is irrotational (A consequence of this is that Bernouilli equation can be applied across streamlines). **(0.5 Point)**
3. Use Bernoulli equation to calculate the pressure at the boundary of the core, assuming that the pressure far away from the tornado ($r \to \infty$) is atmospheric, $p_a = 10^5$ Pa. The air density is $\rho = 1.2$ kg/m^3. **(0.5 Point)**

Hint: The curl operator in cylindrical polar coordinates applied to a vector $\vec{v}$ is:

$$\nabla \times \vec{v} = \left(\frac{1}{r}\frac{\partial v_z}{\partial \theta} - \frac{\partial v_\theta}{\partial z}\right)\vec{e}_r + \left(\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}\right)\vec{e}_\theta + \frac{1}{r}\left(\frac{\partial (r v_\theta)}{\partial r} - \frac{\partial v_r}{\partial \theta}\right)\vec{e}_z.$$

Midterm — Nov. 4th, 2024

Exercise 1:

1) Using continuity:

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0 \Rightarrow$$

$$\frac{\partial v_y}{\partial y} = -\frac{\partial v_x}{\partial x} = -\frac{U_0 y}{2} \frac{e^{-\frac{y}{\sqrt{\nu x / U_0}}}}{\sqrt{\nu x^3 / U_0}} =$$

$$= -\frac{U_0}{2x} \frac{y}{\sqrt{\nu x / U_0}} e^{-\frac{y}{\sqrt{\nu x / U_0}}}.$$

$$\boxed{v_y = v_y(y=0) - \int_0^y \frac{\partial v_x}{\partial y} dy =}$$

$$= -\frac{U_0}{2x} \int_0^y \frac{y}{\sqrt{\nu x / U_0}} e^{-y/\sqrt{\nu x / U_0}} dy =$$

$$= -\frac{U_0}{2x} \sqrt{\nu x / U_0} \int_0^\xi \xi e^{-\xi} d\xi =$$

$$= \frac{U_0}{2} \sqrt{\frac{\nu}{U_0 x}} \left[\left(\frac{y}{\sqrt{\nu x / U_0}} + 1 \right) e^{-\frac{y}{\sqrt{\nu x / U_0}}} - 1 \right]$$

2) The vector normal to the wall is $\vec{n} = \vec{e}_y$. Tangential component:

$$\vec{p}_s = \bar{\bar{\tau}} \cdot \vec{n} = \begin{pmatrix} \tau_{xx} & \tau_{xy} \\ \tau_{xy} & \tau_{yy} \end{pmatrix} \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} = \begin{Bmatrix} \tau_{xy} \\ \tau_{yy} \end{Bmatrix}$$

Tangential.

$$\boxed{\tau_{xy}\Big|_{y=0} = \mu \left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} \right)_{y=0} =}$$

$= 0$, as $v_y(y=0) = 0$ for all x.

$$= \mu \frac{\partial v_x}{\partial y}\Big|_{y=0} + \mu \frac{\partial v_y}{\partial x}\Big|_{y=0} =$$

$$= \mu v_0 \left(\frac{1}{\sqrt{\nu x / v_0}} e^{-\frac{y}{\sqrt{\nu x / v_0}}} \right)_{y=0} =$$

$$= \boxed{\frac{\mu v_0}{\sqrt{\nu x / v_0}}}.$$

3) Normal component:

$$\tau_{yy} = 2\mu \frac{\partial v_y}{\partial y}\bigg|_{y=0} = -2\mu \frac{\partial v_x}{\partial x}\bigg|_{y=0} =$$

$$= -\frac{v_0}{2x}\left|\frac{y}{\sqrt{\nu x / v_0}} e^{-\frac{y}{\sqrt{\nu x / v_0}}}\right|_{y=0} = 0. \;//$$

Exercise 2:

1) At the boundary of the vortex core:

$$v_\theta = \frac{\Gamma}{2\pi R} \Rightarrow \Gamma = 2\pi R v_\theta \approx 2.5 \times 10^{9}\ m^2/s.$$

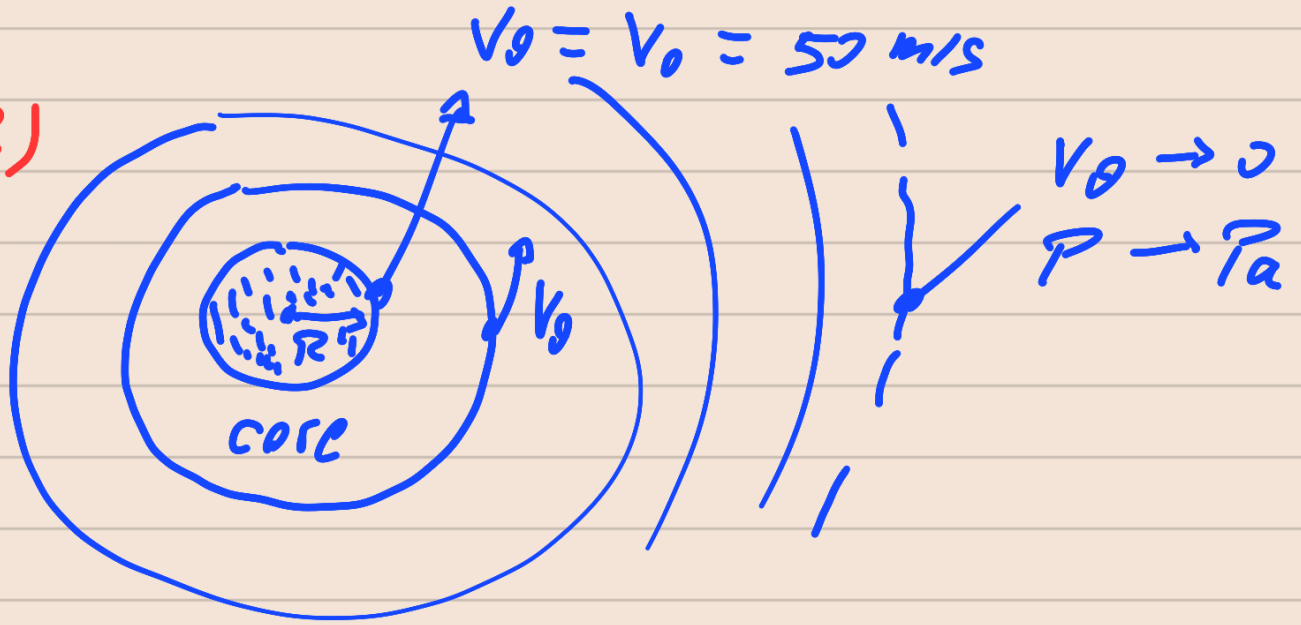
2) Since the flow is 2D, the only non-zero component could be:

$$\omega_z = \frac{1}{r}\left(\frac{\partial (r v_\theta)}{\partial r} - \frac{\partial v_r}{\partial \theta}^{0}\right) =$$

$$= \frac{1}{r}\frac{\partial (r v_\theta)}{\partial r} = \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\Gamma}{2\pi r}\cdot r\right) = 0 \;//$$

So the flow is irrotational!

3)



Applying Bernoulli across streamline:

$$P_a + \frac{1}{2}\rho \cancel{V_\infty^2}^{0} = P_{core} + \frac{1}{2}\rho V_0^2 \Rightarrow$$

$$P_{core} = P_a - \frac{1}{2}\rho V_0^2 =$$

$$= 10^5 Pa - \frac{1}{2} 1.2 \frac{kg}{m^3} \cdot 2500 \frac{m^2}{s^2} =$$

$$= 98.5 \times 10^3 \, Pa$$

FLUID MECHANICS I **03-11-2024**
PROBLEM 1 (5.5 POINTS) 55%. **Time: 70 minutes**

NAME:..... **GRADE:.....**

Jet impinging on a hot deflector plate

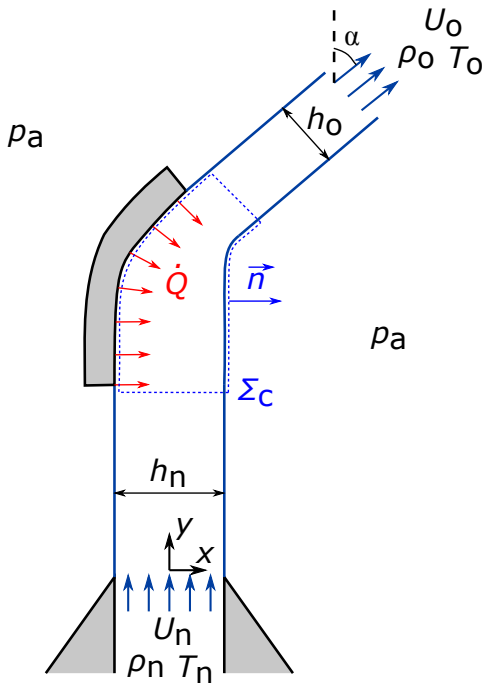
A vertical **two-dimensional** nozzle of width h_n produces a **gas** jet of uniform velocity U_n, density ρ_n, and temperature T_n. The fluid properties in the jet remain constant until it impinges on a hot deflector plate that communicates a **total** heat power $\dot{Q}$ to the fluid, and that deflects its trajectory by an angle α. Upon its contact with the deflector, the jet exits with a new width h_o, which we assume known, and uniform flow parameters (U_o, ρ_o, and T_o) that stay constant as far as we are concerned.

To simplify the calculations, we adopt the following realistic assumptions:

- The pressure at the streamlines that separate the jet from the stagnant outer air is atmospheric, p_a. This means that the pressure inside the yet, when it is not in contact with a solid, is ambient. We also neglect viscous forces on those streamlines.
- The conductive heat flux across these streamlines is also neglected.
- We will not consider body forces.

Our goal is to obtain equations relating the final jet parameters (U_o, ρ_o, and T_o) with the initial ones and any other that you deem necessary. We also wish to obtain the force that the jet exerts on the deflector, $\vec{F}$.

1. Obtain an equation relating the density and temperature in the final state. **(1 Point)**
2. Obtain a relationship between the velocity and density in the final state. **(1 Point)**
3. Obtain another equation relating the temperature, density and velocity in the final state. **(1.5 Point)**
4. Compute the net force exerted on the plate. **(2 Points)**



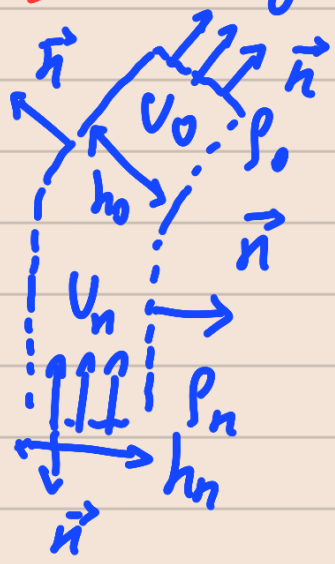
Exercise 3:

1) At sections n and o of the control surface, $P = P_a$, thus using the equation of state:

$$P_a = \rho_n R_g T_n, \quad P_a = \rho_o R_g T_o \Rightarrow$$

$$\Rightarrow \boxed{\rho_n T_n = \rho_o T_o}$$

2) Using mass conservation:



$$\frac{d}{dt}\int_{V_c} \rho\, dv + \int_{\Sigma_c} \rho(\vec{v} - \vec{v_c})\cdot\vec{n}\, d\sigma = 0$$

(steady; $\vec{v_c} = 0$)

$$\int_{\Sigma_{n,o}} \rho \vec{v}\cdot\vec{n}\, d\sigma + \int_{\Sigma_{streamlines}} \rho \vec{v}\cdot\vec{n}\, d\sigma = 0$$

($\vec{v}\cdot\vec{n} = 0$ on streamlines)

$$\int_{\Sigma_n} \rho \vec{v}\cdot\vec{n}\, d\sigma + \int_{\Sigma_o} \rho \vec{v}\cdot\vec{n}\, d\sigma = 0$$

$$-\rho_n v_n h_n + \rho_o v_o h_o = 0$$

$$\boxed{\dot{m} = \rho_n v_n h_n = \rho_o v_o h_o}$$

3) Using energy equation:

$$\frac{d}{dt}\int_{V_c}\rho\left(e+\frac{V^2}{2}\right)dV+\int_{\Sigma_c}\rho\left(e+\frac{V^2}{2}\right)(\vec{V}-\vec{V}_c)\cdot\vec{n}\,d\sigma=$$

$$=-\int_{\Sigma_c}p\vec{V}\cdot\vec{n}\,d\sigma+\int_{\Sigma_c}\vec{V}\cdot\bar{\bar{\tau}}\cdot\vec{n}\,d\sigma+\int_{V_c}\rho\vec{V}\cdot\vec{f}_m\,dV-$$

$$-\int_{\Sigma_c}\vec{q}\cdot\vec{n}\,d\sigma+\int_{V_c}\dot{Q}_V\,dV\Rightarrow$$

$\vec{V}=0$, $\vec{V}=V$ wall, $\bar{\bar{\tau}}=0$ stream lines

$$\int_{\Sigma_i}\rho\left(e+\frac{V^2}{2}\right)(\vec{V}\cdot\vec{n})\,d\sigma=-\int_{\Sigma_e}p\vec{V}\cdot\vec{n}\,d\sigma+\int_{\Sigma_e}\vec{V}\cdot\bar{\bar{\tau}}\cdot\vec{n}\,d\sigma$$

$$-\int_{\Sigma_i}\vec{q}\cdot\vec{n}\,d\sigma\Rightarrow$$

$$-\rho_n V_n\left(c_v T_n+\frac{V_n^2}{2}\right)h_n+\rho_0 V_0\left(c_v T_0+\frac{V_0^2}{2}\right)h_0=$$

$$=p_a V_n h_n-p_a V_0 h_0+\dot{Q},\ \text{using}\ \dot{m}\Rightarrow$$

$$\boxed{\left(c_v T_0+\frac{V_0^2}{2}+\frac{p_a}{\rho_0}\right)-\left(c_v T_n+\frac{V_n^2}{2}+\frac{p_a}{\rho_n}\right)=\frac{\dot{Q}}{\dot{m}}}$$

4) Using momentum conservation:

$$\frac{d}{dt}\int_{V_c}\rho\vec{v}\,d\sigma+\int_{\Sigma_c}\rho\vec{v}(\vec{v}-\vec{v}_c)\cdot\vec{n}\,d\sigma=-\int_{\Sigma_c}(p-p_a)\vec{n}\,d\sigma$$

$$+\int_{\Sigma_c}\vec{\vec{\tau}}'\cdot\vec{n}\,d\sigma+\int_{V_c}\rho\vec{f}_m\,dv\Rightarrow$$

$$-\rho V_n^2 h_n\vec{e}_y+\rho V_0^2 h_0(\sin\alpha\vec{e}_x+\cos\alpha\vec{e}_y)=$$

$$=-\int_{\Sigma_{plate}}(p-p_a)\vec{n}\,d\sigma+\int_{\Sigma_{plate}}\vec{\vec{\tau}}'\cdot\vec{n}\,d\sigma\Rightarrow$$

$$\Rightarrow -\vec{F}_{plate}$$

$$\boxed{\frac{\vec{F}_{plate}}{m}=(V_n-V_0\cos\alpha)\vec{e}_y-V_0\sin\alpha\vec{e}_x}$$