FLUID MECHANICS I **02-11-2021**
PROBLEM 1 (16 POINTS). **Time: 75 minutes**

NAME:..... **GRADE:.....**

The sketch shows a two-dimensional chamber full of liquid of density ρ and viscosity μ. The left and lower walls are fixed while the upper piston moves downwards with a velocity $-dh/dt$ (not necessarily constant), where $h(t)$ is **known** and $h(0) = H$. On the right, the chamber is closed by piston 2 of mass M connected to a spring of constant K and natural length L. This piston, characterized by its position $x_{\rm p}(t)$, is telescopic and adapts its height to the vertical size of the chamber $h(t)$ at the same time that it slides in the horizontal direction without friction with the walls. The full system is immersed in air at pressure p_a.
In the following analysis, use will be made of both the integral conservation equations and the differential conservation equations. With that purpose, two control volumes will be used: the first control volume $\mathcal{V}_c(t)$ is defined as the full chamber containing liquid, while the second control volume $\mathcal{V}_x(t)$ contains the liquid between the left wall and a fixed generic vertical line at x. If gravity is neglected, and taking into account that the **flow is unsteady**,

1. Using the continuity in integral form, determine the relation between $x_{\rm p}(t)$ and $h(t)$ that holds at any instant of time t. Notice that, as shown in fig. 1, $x_{\rm p}(t=0) = L$ and $h(t=0) = H$.

2. Using the mass conservation principle for the volume $\mathcal{V}_x$, and ssuming that the horizontal velocity component v_x is uniform in the y direction, determine the uniform velocity profile $v_x(x,t)$ at a position x.

3. Use $v_x(x,t)$ and the differential mass conservation principle to determine v_y

Using the velocity field $\bar{v} = (v_x, v_y)$ determined above,

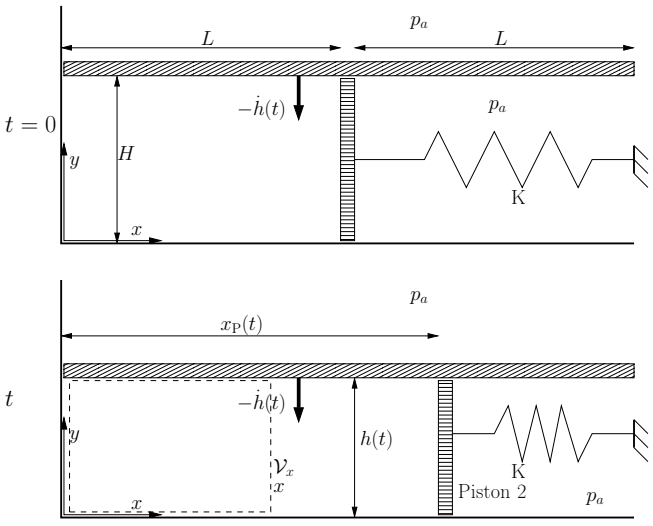
4. Determine the force $\bar{F}$ exerted by the liquid on **all the walls**.

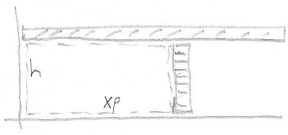
5. Find the pressure field $p(x,y,t)$ that satisfies the momentum equations in differential form. Notice that this pressure can be determined except for an integration constant $C(t)$.

6. Find the force exerted by the liquid on the piston 2.

7. Constant $C(t)$ can be determined if the dynamics of the piston is analyzed. Find $C(t)$.

8. Are the velocity and pressure fields found above solution of the Navier-Stokes equations for this configuration?

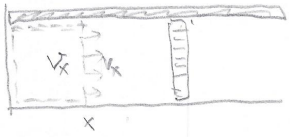




① CONTINUITY V_c:

$$\frac{d}{dt}\int_{V_c}\rho\,dV + \int_{\Sigma_c}\rho(\bar{v}-\bar{v}_c)\cdot\hat{n}\,d\sigma = 0 \Rightarrow$$

$$\frac{d}{dt}\left[x_p \cdot h\right] = 0 \Rightarrow \boxed{x_p \cdot h = L \cdot H}$$



② CONTINUITY V_x:

$$\frac{d}{dt}\int_{V_x}\rho\,dV + \int_{\Sigma_c}\rho(\bar{v}-\bar{v}_c)\cdot\hat{n}\,d\sigma = 0$$

$$\frac{d}{dt}\left[\rho x h(t)\right] + \rho \cdot v_x \cdot h = 0 \rightarrow \dot{x}h + v_x h = 0 \Rightarrow \boxed{v_x = -\frac{\dot{h}}{h}x}$$

③ CONTINUITY (DIFFERENTIAL): $\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0 \rightarrow -\frac{\dot{h}}{h} + \frac{\partial v_y}{\partial y} = 0 \rightarrow \boxed{v_y = +\frac{\dot{h}}{h}y + \cancel{C_1}}$ $\quad v_y(y=0) = 0$, $\quad v_y(y=h) = +\dot{h}$

④ MOMENTUM $\bar{V}_c$:

$$\frac{d}{dt}\int_{V_c}\rho\bar{v}\,dV + \int_{\Sigma_c}\rho\bar{v}(\bar{v}-\bar{v}_c)\cdot\hat{n}\,d\sigma = \underbrace{\int_{\Sigma_c}-p\hat{n}\,d\sigma + \int_{\Sigma_c}\bar{\tau}\cdot\hat{n}\,d\sigma}_{-\bar{F}} + \int_{V_c}\rho\bar{g}\,dV$$

($\int_{\Sigma_c}\rho\bar{v}(\bar{v}-\bar{v}_c)\cdot\hat{n}\,d\sigma = 0$; $\int_{V_c}\rho\bar{g}\,dV = 0$)

x: $\frac{d}{dt}\left[\int_0^{x_p}\rho\left(-\frac{\dot{h}}{h}x\right)h\,dx\right] = -F_x \Rightarrow -\frac{d}{dt}\left[\rho\dot{h}h\cdot\frac{x_p^2}{2}\right] = -F_x$

$$F_x = \frac{\rho\ddot{h}}{2}x_p^2 + \rho\dot{h}x_p\dot{x}_p = \frac{\rho\ddot{h}}{2}\left(\frac{LH}{h}\right)^2 + \rho\dot{h}^2\frac{LH}{h^2}\frac{LH}{h}$$

y: $\frac{d}{dt}\left[\int\rho\frac{\dot{h}}{h}y\,dx\,dy\right] = -F_y \rightarrow \frac{d}{dt}\left[\rho\dot{h}\cdot x_p\cdot\frac{h}{2}\right] = -F_y$

$$\frac{d}{dt}\left(\rho\frac{\dot{h}h}{2}\cdot\frac{LH}{h}\right) = -F_y \rightarrow \boxed{F_y = -\rho\ddot{h}\frac{LH}{2}}$$

⑤ MOMENTUM:

(x): $\rho\frac{dv_x}{dt} + \rho v_x\frac{\partial v_x}{\partial x} + \rho v_y\frac{\partial v_x}{\partial y} = -\frac{\partial p}{\partial x} + \mu\nabla^2 v_x$

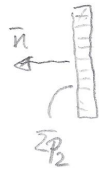
$$-\rho\ddot{h}\frac{x}{h} + \frac{\rho\dot{h}^2}{h^2}x + \rho\dot{h}\frac{x}{h}\cdot\frac{\dot{h}}{h} = -\frac{\partial p}{\partial x} \Rightarrow p = \frac{\rho\ddot{h}}{h}\frac{x^2}{2} - \frac{\rho\dot{h}^2}{h^2}x^2 + f(y)$$

(y): $\rho\frac{dv_y}{dt} + \rho v_x\frac{\partial v_y}{\partial x} + \rho v_y\frac{\partial v_y}{\partial y} = -\frac{\partial p}{\partial y} + \mu\nabla^2 v_y$

$$\rho\frac{\ddot{h}}{h}y - \rho\frac{\dot{h}^2}{h^2}y + \rho\frac{\dot{h}^2}{h^2}y = -\frac{\partial p}{\partial y} = -\frac{df}{dy} \Rightarrow f = -\rho\frac{\ddot{h}}{h}\frac{y^2}{2} + C$$

$$\boxed{p = \frac{\rho\ddot{h}}{h}\left(\frac{x^2-y^2}{2}\right) - \frac{\rho\dot{h}^2}{h^2}x^2 + C}$$

⑥ FORCE ON PISTON:



$$\bar{F} = \int_{\Sigma_{P_2}} -p\bar{n}\,d\sigma + \int_{\Sigma_C} \bar{\bar{\tau}}\cdot\bar{n}\,d\sigma$$

$$\bar{\bar{\tau}}\cdot\bar{n} = \begin{pmatrix} 2\mu\frac{\partial v_x}{\partial x} & 0 \\ 0 & 2\mu\frac{\partial v_y}{\partial y} \end{pmatrix}\cdot\begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} -2\mu\frac{\dot{h}}{h} & 0 \\ 0 & 2\mu\frac{\dot{h}}{h} \end{pmatrix}\begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

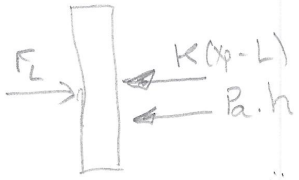
$$= \begin{pmatrix} 2\mu\frac{\dot{h}}{h} \\ 0 \end{pmatrix}$$

$$P(x=x_p,\,y) = \rho\frac{\ddot{h}}{h}\left(x_p^2 - \frac{y^2}{2}\right) - \frac{\rho\dot{h}^2}{h^2}x_p^2 + C$$

$$F = \rho\frac{\ddot{h}}{h}\left(x_p^2\frac{h}{2} - \frac{h^3}{6}\right) - \frac{\rho\dot{h}^2}{h^2}x_p^2 h + Ch + 2\mu\frac{\dot{h}}{h}\cdot h$$

⑦ PISTON DYNAMICS

$$M\frac{d^2x_p}{dt^2} = F_L - K(x_p - L) - P_a\cdot h$$



FROM THIS BALANCE ⇒ C:

$$\dot{x}_p = -\frac{LH}{h^2}\dot{h} \;;\; \ddot{x}_p = LH\left(-\frac{\ddot{h}}{h^2} + 2\frac{\dot{h}^2}{h^3}\right)$$

$$M\cdot LH\cdot\left(-\frac{\ddot{h}}{h^2} + 2\frac{\dot{h}^2}{h^3}\right) = \rho\frac{\ddot{h}}{h}\left(\frac{L^2H^2}{2h^2}h - \frac{h^3}{6}\right) - \rho\frac{\dot{h}^2}{h}\cdot\frac{L^2H^2}{h^2} + 2\mu\dot{h} - K\left(\frac{LH}{h} - L\right) + C\cdot h - P_a\cdot h$$

$$C = P_a + ML\left(-\frac{\ddot{h}}{h^2} + 2\frac{\dot{h}^2}{h^3}\right) - \rho\frac{\ddot{h}}{h^2}\left(\frac{L^2H^2}{2h} - \frac{h^3}{6}\right) + \rho\frac{\dot{h}^2}{h^2}\cdot\frac{L^2H^2}{h^2} - 2\mu\frac{\dot{h}}{h} + K\left(\frac{LH}{h} - L\right)$$

⑧ NO. THEY SATISFY DIFFERENTIAL EQNS. BUT NOT THE NO-SLIP BOUNDARY CONDITIONS.

FLUID MECHANICS I **02-11-2021**
EXERCISE 2 (4 POINTS). **Time: 15 minutes**

NAME:..... **GRADE:.....**

Consider the momentum equation in differential form $D\bar{v}/Dt = -\nabla p + \nabla \cdot \bar{\bar{\tau}}' + \rho \bar{f}_m$. Prove that, for a perfect liquid of constant viscosity μ, the divergence of the stress tensor $\nabla \cdot \bar{\bar{\tau}}'$ is equal to $\mu \nabla^2 \bar{v}$. You only need to consider the 2D case in cartesian coordinates.

Note:

$$\bar{\bar{\tau}}' = \begin{pmatrix} \tau'_{xx} & \tau'_{xy} \\ \tau'_{xy} & \tau'_{yy} \end{pmatrix} + (\mu_B - \frac{2}{3}\mu)\nabla \cdot \bar{v}\bar{\bar{I}}$$

where $\tau'_{xx} = 2\mu \dfrac{\partial v_x}{\partial x}$, $\tau'_{xy} = \mu \left(\dfrac{\partial v_x}{\partial y} + \dfrac{\partial v_y}{\partial x} \right)$ and $\tau'_{yy} = 2\mu \dfrac{\partial v_y}{\partial y}$

LIQUID $\Rightarrow \nabla\cdot\vec{v}=0 \Rightarrow$ $\bar{\bar{\tau}}' = \begin{pmatrix} \tau'_{xx} & \tau'_{xy} \\ \tau'_{yx} & \tau'_{yy} \end{pmatrix}$; $\nabla\cdot\vec{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0$ (1)

$$\nabla\cdot\bar{\bar{\tau}}' = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)\cdot\begin{pmatrix} \tau'_{xx} & \tau'_{xy} \\ \tau'_{yx} & \tau'_{yy} \end{pmatrix} = \left(\underbrace{\frac{\partial \tau'_{xx}}{\partial x} + \frac{\partial \tau'_{xy}}{\partial y}}_{\text{I}}, \underbrace{\frac{\partial \tau'_{xy}}{\partial x} + \frac{\partial \tau'_{yy}}{\partial y}}_{\text{II}}\right)$$

$$\text{I} = 2\mu\frac{\partial^2 v_x}{\partial x^2} + \mu\frac{\partial^2 v_x}{\partial y^2} + \mu\frac{\partial^2 v_y}{\partial x \partial y} = 2\mu\frac{\partial^2 v_x}{\partial x^2} + \mu\frac{\partial^2 v_x}{\partial y^2} - \mu\frac{\partial^2 v_x}{\partial x^2} = \mu\nabla^2 v_x$$

$$\uparrow \quad \frac{\partial(1)}{\partial x} = \frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_y}{\partial x \partial y}$$

$$\text{II} = \mu\frac{\partial^2 v_x}{\partial x \partial y} + \mu\frac{\partial^2 v_y}{\partial x^2} + 2\mu\frac{\partial^2 v_y}{\partial y^2} = -\mu\frac{\partial^2 v_y}{\partial y^2} + \mu\frac{\partial^2 v_y}{\partial x^2} + 2\mu\frac{\partial^2 v_y}{\partial y^2} = \mu\nabla^2 v_y$$

$$\uparrow \quad \frac{\partial(1)}{\partial y} = \frac{\partial^2 v_x}{\partial x \partial y} + \frac{\partial^2 v_y}{\partial y^2}$$

THEREFORE:

$$\underline{\nabla\cdot\bar{\bar{\tau}}' = \mu\nabla^2\vec{v}}$$