**FLUID MECHANICS I** **10-11-2020**
**PROBLEM 1 (12 POINTS).** **Time: 75 minutes**

**NAME:.....** **GRADE:.....**

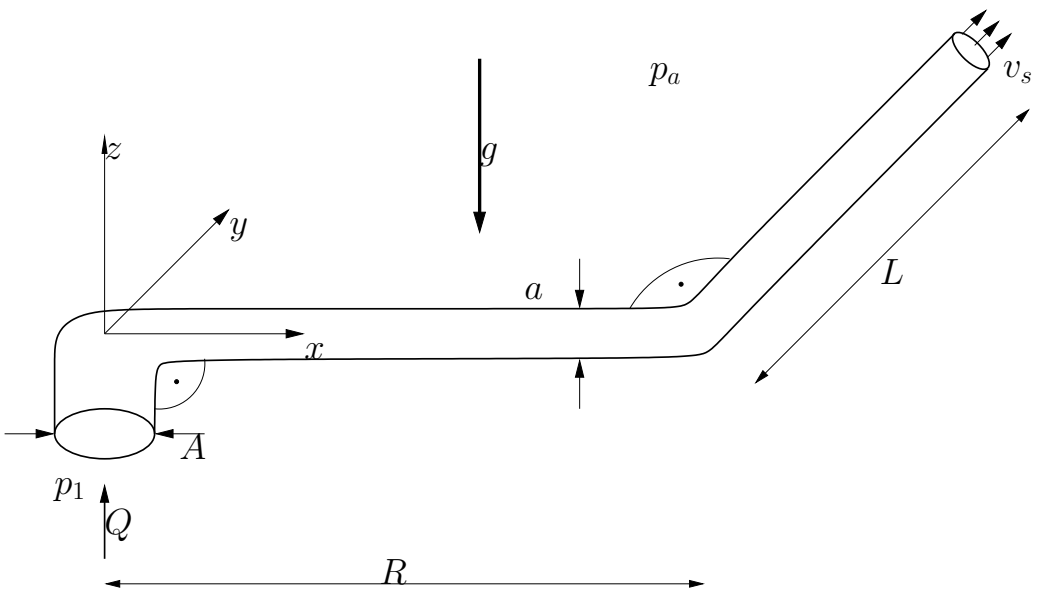
The sketch shows a double-elbow pipe. A flow rate of water $Q$, at a pressure $p_1$, enters the system upwards through a pipe of area $A$, the flow rate then travels through a piece of pipe of length $R$ aligned with the $x$-axis and, after a $90^o$ curve, the water discharges, aligned with the $y$-axis, in the open atmosphere.

1. Determine the discharge velocity $v_s$ of the water in the atmosphere
2. Find the net force, exerted by **both** the liquid and the atmosphere, on the pipe walls
3. Determine the net torque exerted on the pipe

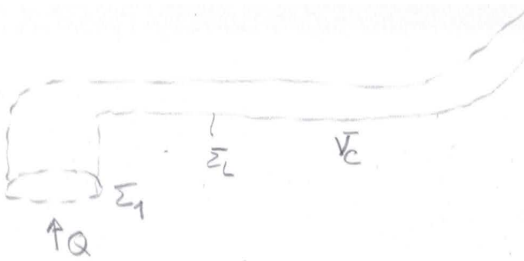
Next, an analysis is needed of the transient process of filling the pipe when it is initially empty. In this case, we will assume the water-air interface to be flat and perpendicular to the direction of the movement. The interface advances through the pipe until the exit is reached.

4. Determine the time $t_1$ needed for the water to reach the second elbow, and the time $t_2$ to reach the exit
5. For $t_1 < t < t_2$, sketch the position of the water interface and show that, along each of the pieces of pipe, the velocity of the liquid is uniform
6. In the same time interval, determine the net force exerted on the pipe
7. **(optional: +1 point)** In the same time interval, determine the net torque exerted on the pipe

Take into account the gravity. Consider that the force exerted by the air is due only to the pressure, which can be taken as uniform. Neglect the volumes associated with with the elbows regions.



①




CONTINUITY: $\frac{d}{dt}\int_{V_C}\rho\,dV + \int_{\Sigma_C}\rho(\bar{v}-\bar{v}_c)\cdot\bar{n}\,d\sigma = 0 \rightarrow -Q + V_S\cdot\pi a^2 = 0 \rightarrow \boxed{V_S = \frac{Q}{\pi a^2}}$

② MOMENTUM: $\frac{d}{dt}\int_{V_C}\rho\bar{v}\,dV + \int_{\Sigma_C}\rho\bar{v}(\bar{v}-\bar{v}_c)\cdot\bar{n}\,d\sigma = \int_{\Sigma_L}-p\bar{n}\,d\sigma + \int_{\Sigma_L}\bar{\bar{\tau}}'\cdot\bar{n}\,d\sigma + \int_{V_C}\rho\bar{f}_m\,dV$

$-\rho\frac{Q^2}{A}\bar{e}_z + \rho\frac{Q^2}{\pi a^2}\bar{e}_y = P_1\cdot A\,\bar{e}_z - P_a\cdot\pi a^2\bar{e}_y - \rho g\pi a^2(R+L)\bar{e}_z - \bar{F}_L$

$\bar{F}_L = \left[P_1 A + \rho\frac{Q^2}{A} - \rho g\pi a^2(R+L)\right]\bar{e}_z - \left[P_a\pi a^2 + \rho\frac{Q^2}{\pi a^2}\right]\bar{e}_y$

$\bar{F}_A = \underbrace{\int_{\Sigma_L}-P_a\bar{n}\,d\sigma}_{=0} + \underbrace{\int_{\Sigma_1+\Sigma_2}-P_a\bar{n}}_{=0} - \int_{\Sigma_4+\Sigma_5}-P_a\bar{n}\,d\sigma = \int_{\Sigma_4+\Sigma_5}P_a\cdot\bar{n}\,d\sigma = -P_a A\bar{e}_z + P_a\pi a^2\bar{e}_y$

$\boxed{\bar{F} = \bar{F}_L + \bar{F}_A = \left[(P_1-P_a)A + \rho\frac{Q^2}{A} - \rho g\pi a^2(L+R)\right]\bar{e}_z - \rho\frac{Q^2}{\pi a^2}\bar{e}_y}$

③ ANGULAR MOMENTUM. (ABOUT ORIGIN).

$\frac{d}{dt}\int_{V_C}\bar{x}_A\wedge\rho\bar{v}\,dV + \int_{\Sigma_C}\bar{x}_A\wedge\rho\bar{v}(\bar{v}-\bar{v}_c)\cdot\bar{n}\,d\sigma = \int_{\Sigma_L}\bar{x}_A\wedge(-P\bar{n})d\sigma + \int_{\Sigma_L}\bar{x}_A\wedge(\bar{\bar{\tau}}'\cdot\bar{n})d\sigma + \int_{V_C}\bar{x}_A\wedge\rho\bar{f}_m\,dV$

$= -\bar{M}_L$

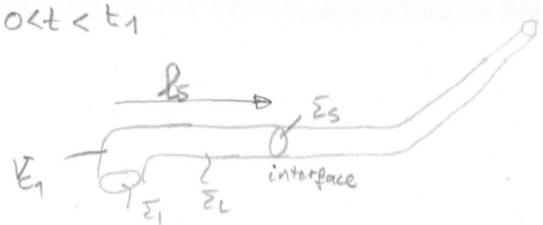
$\rho R\frac{Q}{A}Q\bar{e}_z = -RP_a\pi a^2\bar{e}_z - \bar{M}_L + \rho g\pi a^2\frac{R^2}{2}\bar{e}_y + \rho g\pi a^2 RL\bar{e}_y - \rho g\pi a^2\frac{L^2}{2}\bar{e}_x$

$\bar{M}_L = \left(-\rho R\frac{Q^2}{A} - RP_a\pi a^2\right)\bar{e}_z + \rho g\pi a^2\left(\frac{R^2}{2}+RL\right)\bar{e}_y - \rho g\pi a^2\frac{L^2}{2}\bar{e}_x$

$\bar{M}_A = \underbrace{\int_{\Sigma_L}\bar{x}_A(-P_a\bar{n})\,d\sigma + \int_{\Sigma_1+\Sigma_2}\bar{x}_A(-P_a\bar{n})d\sigma}_{=0} - \int_{\Sigma_4+\Sigma_5}\bar{x}_A(-P_a\bar{n})d\sigma = RP_a\pi a^2\bar{e}_z$

$\boxed{\bar{M} = \bar{M}_L + \bar{M}_A = -\rho RQ\frac{Q}{A}\bar{e}_z + \rho g\pi a^2\left(\frac{R^2}{2}+RL\right)\bar{e}_y - \rho g\pi a^2\frac{L^2}{2}\bar{e}_x}$

④ $0 < t < t_1$



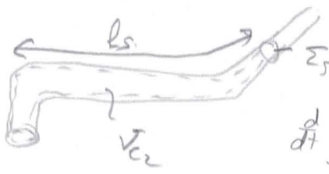

CONTINUITY:

$$\frac{d}{dt}\int_{V_c} \rho\, dV + \int_{\Sigma_c} \rho(\vec{v}-\vec{v}_c)\cdot\vec{n}\, d\sigma$$

$$\frac{d}{dt}[\rho \cdot \pi a^2 l_s] - \rho Q = 0 \rightarrow \dot{l}_{s_1} = \frac{Q}{\pi a^2}\,; \quad l_{s_1} = \frac{Q}{\pi a^2} t$$

$$l_s = R \Rightarrow \boxed{t_1 = \frac{\pi a^2 R}{Q}}$$

$t_1 < t < t_2$:

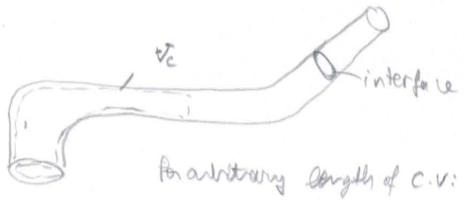



$$\frac{d}{dt}\int_{V_c} \rho\, dV + \int_{\Sigma_c} \rho(\vec{v}-\vec{v}_c)\cdot\vec{n}\, d\sigma = 0$$

$$\frac{d}{dt}[\rho \pi a^2 l_s] - \rho Q = 0 \quad \dot{l}_s = \frac{Q}{\pi a^2}$$

$$l_s = R + L \Rightarrow \boxed{t_2 = \frac{\pi a^2 (L+R)}{Q}}$$

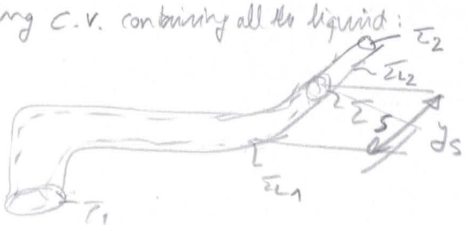
⑤




For arbitrary length of C.V.: $\frac{d}{dt}\int \rho\, dV + \int_{\Sigma_c} \rho(\vec{v}-\vec{v}_c)\cdot\vec{n}\, d\sigma = 0$

$$-Q + V\pi a^2 = 0 \Rightarrow V = \frac{Q}{\pi a^2}$$ along any position along the pipe

⑥ Using C.V. containing all the liquid:




$$\frac{d}{dt}\int_{V_c} \rho\vec{v}\, dV + \int_{\Sigma_c} \rho\vec{v}(\vec{v}-\vec{v}_c)\cdot\vec{n}\, d\sigma = \int_{\Sigma_c} -p\vec{n}\, d\sigma + \int_{\Sigma_c} \vec{\tau}\cdot\vec{n}\, d\sigma + \int_{V_c} \rho\vec{g}\, dV$$

$$\frac{d}{dt}\left[\rho \frac{Q}{\pi a^2}\cdot \pi a^2 R\,\vec{e}_x + \rho \cdot \frac{Q}{\pi a^2}\pi a^2 y_s \vec{e}_y\right] - \rho\frac{Q^2}{A}\vec{e}_z = P_1 \cdot A\vec{e}_z - P_a \cdot \pi a^2 \vec{e}_y - \vec{F}_L - \rho g(\pi a^2 (R + y_s))\vec{e}_z$$

$$\vec{F}_L = -\rho Q \dot{y}_s \vec{e}_y + \rho\frac{Q^2}{A}\vec{e}_z + P_1 A\vec{e}_z - P_a \pi a^2 \vec{e}_y - \rho g \pi a^2 (R + y_s)\vec{e}_z$$

$$\vec{F}_A = \underbrace{\int_{\Sigma_{L_1}+\Sigma_{L_2}} -P_a \vec{n}\, d\sigma}_{=0} + \int_{\Sigma_2+\Sigma_1} -P_a \vec{n}\, d\sigma - \int_{\Sigma_1+\Sigma_2} -P_a \vec{n}\, d\sigma \quad ; \quad \dot{y}_s = v_s = \frac{Q}{\pi a^2}\,; \; y_s = \frac{Q}{A}(t - t_1)$$

$$\bar{F} = \bar{F}_L + \bar{F}_A = \left[ \rho \frac{Q^2}{A} + (P_1 - P_2) A \right] \bar{e}_z - \rho g \pi a^2 (R + y_s) \bar{e}_z$$

$$- \rho \frac{Q^2}{\pi a^2} \bar{e}_y$$

(7) ANGULAR MOMENTUM . SAME C.V.

$$\frac{d}{dt} \int_{V_c} \bar{x}_A \rho \bar{v} \, dV + \int_{\Sigma_c} \bar{x}_A \rho \bar{v} (\bar{v} - \bar{v}_c) \cdot \bar{n} \, d\sigma = \int_{\Sigma_c} \bar{x}_A (-p \bar{n}) d\sigma + \int_{\Sigma_c} \bar{x}_A (\bar{\tau} \cdot \bar{n}) d\sigma + \int_{V_c} \bar{x}_A \rho \bar{f}_m dV$$

$$\frac{d}{dt} \left[ \rho \dot{y}_s R \pi a^2 y_s \right] \bar{e}_z + 0 = - R P_a \pi a^2 \bar{e}_z - \bar{M}_L + \rho g \pi a^2 \frac{R^2}{2} \bar{e}_y + \rho g \pi a^2 R y_s \bar{e}_y - \rho g \pi a^2 \frac{y_s^2}{2} \bar{e}_x$$

$$\bar{M}_L = - \rho g \pi a^2 \frac{y_s^2}{2} \bar{e}_x + \rho g \pi a^2 \left( \frac{R^2}{2} + R y_s \right) \bar{e}_y + \left[ - \rho \pi a^2 R \dot{y}_s^2 - R P_a \pi a^2 \right] \bar{e}_z$$

$$\bar{M}_A = \underbrace{\int_{\Sigma_{L_1} + \Sigma_{L_2}} \bar{x}_A (-P_a \bar{n}) d\sigma}_{0} + \int_{\Sigma_1 + \Sigma_2} \bar{x}_A (-P_a \bar{n}) d\sigma - \int_{\Sigma_1 + \Sigma_2} \bar{x}_A (-P_a \bar{n}) d\sigma$$

$$\boxed{\bar{M} = \bar{M}_L + \bar{M}_A = - \rho g \pi a^2 \frac{y_s^2}{2} \bar{e}_x + \rho g \pi a^2 \left( \frac{R^2}{2} + R y_s \right) \bar{e}_y - \rho \pi a^2 R \dot{y}_s^2 \bar{e}_z}$$

WITH $\dot{y}_s = \frac{Q}{\pi a^2}$

$y_s = \frac{Q}{\pi a^2} (t - t_1)$

**FLUID MECHANICS I** **10-11-2020**
**PROBLEM 2 (8 POINTS).** **Time: 35 minutes**

**NAME:.....** **GRADE:.....**

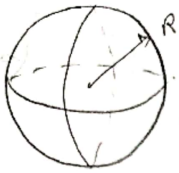
A solid sphere is immersed in a liquid with density $\rho$ and viscosity $\mu$. The radius of the sphere changes periodically as $R(t) = R_0(1 + \epsilon \sin(\omega t))$ with $R_0$, $\epsilon \ll R_0$ and $\omega$ known constants.

1. Obtain the velocity field induced by the sphere in the liquid.

2. If the pressure of the liquid is $p = p_\infty$ at $t = 0$, obtain the pressure field giving, in particular, the pressure of the liquid on the surface of the sphere.

3. Write the equation, with boundary and initial conditions, that would allow the calculation of the temperature field if the liquid is initially at temperature $T_\infty$ and the sphere is heated up to reach a constant temperature $T_0 > T_\infty$ at $t = 0$. The thermal conductivity $k$ and the specific heat of the fluid $c$ are known.

4. Simplify and solve previous equation to obtain the temperature field considering that viscous dissipation is negligible and assuming that the problem is quasisteady $R_0^2\omega/(k/\rho c) \ll 1$.



$R(t) = R_0 \cdot (1 + \varepsilon \cdot \cos(\omega t))$

$f_\infty, T_\infty$
$\rho, \mu, c, k$

1) Continuity equation: $\frac{\partial \rho}{\partial t} + \nabla(\rho \vec{v}) = 0 \Rightarrow r^2 v_r = C_1$ $\Big\}$ $\boxed{v_r = \frac{R^2(t)}{r^2} \cdot \dot{R}}$

At $r = R(t) \Rightarrow v_r = \dot{R} = R_0 \varepsilon \omega \cdot \cos(\omega t)$

2) $\rho \frac{\partial v_r}{\partial t} + \rho v_r \cdot \frac{\partial v_r}{\partial r} = -\frac{\partial p}{\partial r} + \mu \frac{\partial}{\partial r}\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 v_r\right)\right)$ (the viscous term $\to 0$)

$$\rho \cdot \frac{2R\dot{R}^2}{r^2} + \rho \frac{R^2}{r^2}\ddot{R} - 2\rho \frac{R^2\dot{R}}{r^3} \cdot \frac{R^2\dot{R}}{r^3} = -\frac{\partial p}{\partial r}$$

$$p(r,t) = -\int \left( 2\rho \frac{R}{r^2}\dot{R}^2 + \rho\left(\frac{R}{r}\right)^2 \ddot{R} - 2\rho \frac{R^4 \dot{R}^2}{r^5} \right) dr + C$$

$$p - p_\infty = -\left[ -2\rho \frac{R}{r}\dot{R}^2 - \rho \frac{R^2}{r}\ddot{R} + \frac{1}{2}\rho \frac{R^4\dot{R}^2}{r^4} \right] = 2\rho \frac{R}{r}\dot{R}^2 \left[ 1 - \frac{1}{2} R \frac{\ddot{R}}{\dot{R}^2} \right] + \frac{\rho}{2}\frac{R^4\dot{R}^2}{r^4}$$

At $r = R$

$$p_s - p_\infty = 2\rho \dot{R}^2 + \rho R \ddot{R} - \frac{1}{2}\rho \dot{R}^2$$

$$p_s - p_\infty = \frac{3}{4}\rho \dot{R}^2 + \rho R \ddot{R}$$

3) $\begin{cases} \rho \cdot c \cdot \frac{\partial T}{\partial t} + \rho c \cdot v_r \cdot \frac{\partial T}{\partial r} = \phi_v + \frac{k}{r^2} \cdot \frac{\partial}{\partial r}\left(r^2 \frac{\partial T}{\partial r}\right) \\ \phi_v = \mu \cdot \left\{ 2 \cdot \left[ \left(\frac{\partial v_r}{\partial r}\right)^2 + 2\left(\frac{v_r}{r}\right)^2 \right] \right\} \end{cases}$

$$\boxed{\begin{aligned} &\frac{\partial T}{\partial t} + \left(\frac{R}{r}\right)^2 \dot{R} \cdot \frac{\partial T}{\partial r} = \frac{12\mu}{\rho c} \cdot \frac{R^4}{r^6}\dot{R}^2 + \frac{k/\rho c}{r^2} \cdot \frac{\partial}{\partial r}\left(r^2 \frac{\partial T}{\partial r}\right) \\ &T(r = R) = T_0 \\ &T(r = \infty) = T_\infty \end{aligned}}$$

4) Conduction time $t_k \sim \frac{R_0^2}{(k/\rho c)}$ $\Big\}$ $\frac{t_k}{t_c} \sim \frac{\omega R_0^2}{(k/\rho c)} \ll 1$

Characteristic time $t_c \sim \omega^{-1}$

$\phi_v \sim \mu \cdot \omega_0^2$

$\frac{k/\rho c}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial T}{\partial r}\right) \sim \frac{k}{\rho c}\frac{(T_0 - T_\infty)}{R_0^2}$ $\Big\}$ $\frac{\mu \cdot \omega_0^2}{\frac{k}{\rho c}\frac{(T_0 - T_\infty)}{R_0^2}} \ll 1$

$$\frac{R^2}{r^2}\dot{R}\frac{\partial T}{\partial r} = \frac{k/\rho c}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial T}{\partial r}\right)$$

$$\boxed{\frac{T - T_\infty}{T_0 - T_\infty} = \exp\left\{ -\frac{\dot{R}R^2}{k/\rho c}\left(\frac{1}{r} - \frac{1}{R}\right) \right\}}$$