INTRODUCTION TO FLUID MECHANICS. **06-11-2017**
PROBLEM 1 (10 POINTS). **Time: 90 minutes**

NAME:..... **GRADE:.....**

The figure below is a sketch of a 2-dimensional flowmeter. The liquid, of density ρ, enters the system through 2 slots with velocities V_0 at angles $\pm 45^o$ with the vertical, pressure p_0 and temperatures T_0 and $T_0 + \Delta T$ –see the sketch–. The position of the wedge of mass M is a function of the flow rate, once a steady state has been reached. The system operates and discharges in the atmosphere at pressure p_a and under normal gravity conditions. The volume of the cavity, excluding the wedge, is $\mathcal{V}$. If the system is symmetric about a vertical axis at the center of the figure,

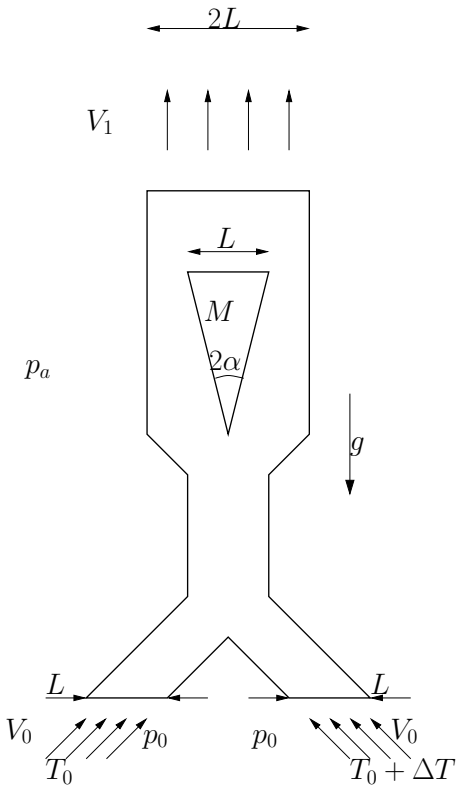
1. For the analysis to be carried out below, define **clearly** an adequate control volume. Make an sketch and specify its boundaries.

2. Determine the velocity V_1 at the exit.

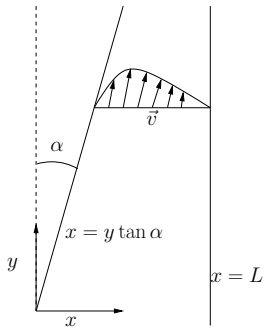
3. Find the force exerted by the liquid on the wedge $\bar{F}_{L\to\text{wedge}}$.

4. Find the forces exerted by the liquid and the atmosphere on the outer case, $\bar{F}_{L\to\text{case}}$ and $\bar{F}_{A\to\text{case}}$, respectively.

5. Determine the heat transfer to the liquid through the casing q_L if the temperature of the fluid at the exit is T_0. Can the heat flux to the wedge be determined at equilibrium ($t \to \infty$)?

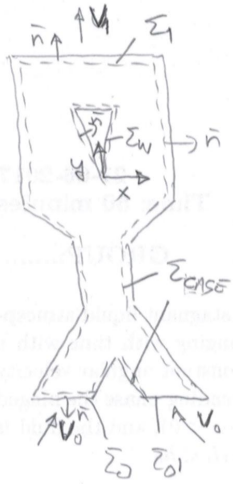


In the slender region between the wedge and the outer case, we look for a local solution of the Navier-Stokes equations with a y-component given by $v(x,y) = V(y)(\bar{x} - \bar{y}\tan\alpha)(1-\bar{x})$, where $\bar{x} = x/L$ and $\bar{y} = y/L$,



6. Use an adequate control volume to determine $V(y)$, so that the global continuity equation is satisfied.

7. Obtain the values of α for which the above expression for $v(x,y)$ is part of an exact solution of the Navier-Stokes equations and find the solution in such cases. In particular determine the x-velocity component $u(x,y)$ and pressure $p(x,y)$. (**2 extra points**)

① 

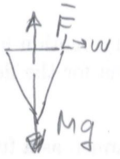
② Continuity:

$$\frac{d}{dt}\int_{V_c}\rho\,dV + \int_{\Sigma_c}\rho(\bar{v}-\bar{v}_c)\cdot\bar{n}\,d\sigma = 0$$

Σ_c: Σ_0, $\Sigma_{0'}$, Σ_1, Σ_{CASE}, Σ_W

$$-\rho\frac{U_0}{\sqrt{2}}L - \rho\frac{U_0}{\sqrt{2}}L + \rho V_1\,2L = 0 \rightarrow \boxed{V_1 = \frac{U_0}{\sqrt{2}}}$$

③ WEDGE IN EQUILIBRIUM:



$$\bar{F}_{L-W} - Mg\bar{e}_y = 0 \quad \boxed{\bar{F}_{L\to W} = Mg\bar{e}_y}$$

④ C.V ANALYSIS: MOMENTUM:

$$\frac{d}{dt}\int_{V_c}\rho\bar{v}\,dV + \int_{\Sigma_c}\rho\bar{v}(\bar{v}-\bar{v}_c)\cdot\bar{n}\,d\sigma = \int_{\Sigma_c}-p\bar{n}\,d\sigma + \int_{\Sigma_c}\bar{\bar{\tau}}'\cdot\bar{n}\,d\sigma + \int_{V_c}-\rho g\bar{e}_y\,dV$$

STEADY; $\bar{v}=\bar{v}_c=0$ on Σ_{CASE}; Σ_c: Σ_0, $\Sigma_{0'}$, Σ_1, Σ_W, Σ_{CASE}; $\Sigma_W \rightarrow -\bar{F}_{L\to W}$; $\Sigma_{CASE} \Rightarrow -\bar{F}_{L\to CASE}$

$$-\frac{V_0}{\sqrt{2}}L\rho\left(\frac{V_0}{\sqrt{2}}\bar{e}_x + \frac{V_0}{\sqrt{2}}\bar{e}_y\right) - \frac{V_0}{\sqrt{2}}L\rho\left(\frac{V_0}{\sqrt{2}}\bar{e}_x + \frac{V_0}{\sqrt{2}}\bar{e}_y\right) = \underbrace{P_0 L\bar{e}_y}_{\Sigma_0} + \underbrace{P_0 L\bar{e}_y}_{\Sigma_{0'}} \underbrace{-2P_a L\bar{e}_y}_{\Sigma_1} - \bar{F}_{L\to W} - \rho g\forall\bar{e}_y - \bar{F}_{L\to CASE}$$

$$\boxed{\bar{F}_{L\to CASE} = \left\{\left[\rho V_0^2\frac{L}{2} + (P_0 - P_a)\right]2L - \rho g\forall - Mg\right\}\bar{e}_y}$$

$$\boxed{\bar{F}_{A\to CASE} = \underbrace{\int_{\Sigma_{CASE}} -P_a\bar{n}\,d\sigma + \int_{\Sigma_0+\Sigma_{0'}+\Sigma_1} -P_a\bar{n}\,d\sigma}_{=0} - \int_{\Sigma_0+\Sigma_{0'}+\Sigma_1} -P_a\bar{n}\,d\sigma = \int_{\Sigma_0+\Sigma_{0'}+\Sigma_1} P_a\bar{n}\,d\sigma = \left(P_a\cdot 2L - P_a\cdot 2L\right)\bar{e}_y = 0}$$

⑤ INTERNAL ENERGY EQUATION:

$$\frac{d}{dt}\int_{V_c}\rho e\,dV + \int_{\Sigma_c}\rho e(\bar{v}-\bar{v}_c)\cdot\bar{n}\,d\sigma = \int_{\Sigma_c}-\bar{q}\cdot\bar{n}\,d\sigma + \int_{V_c}(\dot{Q}_r+\dot{Q}_{ch})\,dV + \left[\int_{V_c}-P\nabla\cdot\bar{v}\,dV + \int_{V_c}\phi_v\,dV\right]$$

Σ_c: Σ_0, $\Sigma_{0'}$, Σ_1, Σ_W, Σ_{CASE}; $\dot{Q}_r = \dot{Q}_{ch} = 0$

$$-\rho C\frac{U_0}{\sqrt{2}}T_0 - \rho C\frac{U_0}{\sqrt{2}}(T_0+\Delta T) + \rho C\frac{U_0}{\sqrt{2}}\cdot 2LT_0 = \dot{q}_L + \int_{V_c}\phi_v\,dV$$

REMOVE EXCESS T AND DISSIPATION

$$\boxed{\dot{q}_L = -\rho C\frac{U_0}{\sqrt{2}}\Delta T - \int_{V_c}\phi_v\,dV} \quad ; \quad \int_{\Sigma_W}\bar{q}\cdot\bar{n}\,d\sigma = 0 \text{ at equilibrium.}$$

⑥



$$v(x,y) = V(y)(\bar{x} - \bar{y}\tg\alpha)(1-\bar{x})$$

$$\int_{x=y\tg\alpha}^{x=L} \bar{v}\cdot\bar{n}\,d\sigma = \frac{V_0}{\sqrt{2}}\cdot L$$

$$V(y)\int_{\bar{x}=\bar{y}\tg\alpha}^{\bar{x}=1} (\bar{x}-\bar{y}\tg\alpha)(1-\bar{x})\,d\bar{x} = \frac{V_0}{\sqrt{2}}$$

$$V(y)\left[-\frac{\bar{x}^3}{3} + (\bar{y}\tg\alpha+1)\frac{\bar{x}^2}{2} - \bar{y}\tg\alpha\,\bar{x}\right]_{\bar{x}=\bar{y}\tg\alpha}^{\bar{x}=1} = \frac{V_0}{\sqrt{2}}$$

$$\boxed{V(y) = \frac{V_0}{\sqrt{2}}\cdot\frac{1}{\left[\frac{\bar{y}^3\tg^3\alpha - 1}{3} + (\bar{y}\tg\alpha+1)\left(\frac{1-\bar{y}^2\tg^2\alpha}{2}\right) - \bar{y}\tg\alpha(1-\bar{y}\tg\alpha)\right]}}$$

⑦ CONTINUITY (DIFFERENTIAL):

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \rightarrow \int_{x=y\tg\alpha}^{x} \frac{\partial u}{\partial x}\,dx = -\int_{x=y\tg\alpha}^{x} \frac{\partial v}{\partial y}\,dx \rightarrow u - \underbrace{u(x=y\tg\alpha, y)}_{0 \text{ NO-SLIP}} = -\int_{x=y\tg\alpha}^{x} \frac{\partial v}{\partial y}\,dx$$

For $x=L$: $u=0 \rightarrow$

$$0 = -\int_{x=y\tg\alpha}^{x=L} \frac{\partial v}{\partial y}\,dx$$

if $v = V(y)\cdot(\bar{x}-\bar{y}\tg\alpha)(1-\bar{x}) \rightarrow \frac{\partial v}{\partial y} = \frac{V'}{L}(\bar{x}-\bar{y}\tg\alpha)(1-\bar{x}) - \frac{V}{L}\tg\alpha(1-\bar{x})$

$$\int_{\bar{x}=\bar{y}\tg\alpha}^{1} \frac{\partial v}{\partial y}\,d\bar{x} = 0 \rightarrow \frac{V'(y)}{L}\left[-\frac{\bar{x}^3}{3} + (\bar{y}\tg\alpha+1)\frac{\bar{x}^2}{2} - \bar{y}\tg\alpha\,\bar{x}\right]_{\bar{x}=\bar{y}\tg\alpha}^{\bar{x}=1} - \frac{V}{L}\tg\alpha\left(\bar{x}-\frac{\bar{x}^2}{2}\right)\Big|_{\bar{x}=\bar{y}\tg\alpha}^{1} = 0$$

$$0 = \frac{V'(y)}{L}\underbrace{\left[\frac{\bar{y}^3\tg^3\alpha}{3} - 1 + (\bar{y}\tg\alpha+1)\left(1-\frac{\bar{y}^2\tg^2\alpha}{2}\right) - \bar{y}\tg\alpha(1-\bar{y}\tg\alpha)\right]}_{\frac{V_0}{V(y)\cdot\sqrt{2}}} - \frac{V}{L}\tg\alpha\left(\frac{1}{2} - \bar{y}\tg\alpha + \frac{\bar{y}^2\tg^2\alpha}{2}\right)$$

$$\frac{dV}{dy}\cdot\frac{V_0}{V(y)\sqrt{2}} - V(y)\tg\alpha\left(1 - \bar{y}\tg\alpha + \frac{\bar{y}^2\tg^2\alpha}{2}\right) = 0$$

$$\frac{dV}{V^2} = \frac{\sqrt{2}}{V_0}\tg\alpha\left(1 - \bar{y}\tg\alpha + \frac{\bar{y}^2\tg^2\alpha}{2}\right)dy$$

$$-\frac{1}{V} = \frac{\sqrt{2}}{V_0}\tg\alpha\left(1 - \frac{2}{2}\bar{y}\tg\alpha + \frac{\bar{y}^3\tg^2\alpha}{6}\right) + C$$

this is consistent with the above expression if $\alpha=0 \Rightarrow$ $\boxed{V = \text{const} = \frac{6V_0}{\sqrt{2}}}$ $\rightarrow \frac{\partial u}{\partial x} = 0 \rightarrow \boxed{u=0}$

Momentum x: $\frac{\partial p}{\partial x} = 0 \rightarrow p = p(y)$

Momentum y: ~~$\rho\frac{\partial v}{\partial t} + \rho u\frac{\partial v}{\partial x} + \rho v\frac{\partial v}{\partial y}$~~ $= -\frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) - \rho g \rightarrow$ $\boxed{p + \rho g y = \text{const.}}$