ESCUELA POLITÉCNICA SUPERIOR

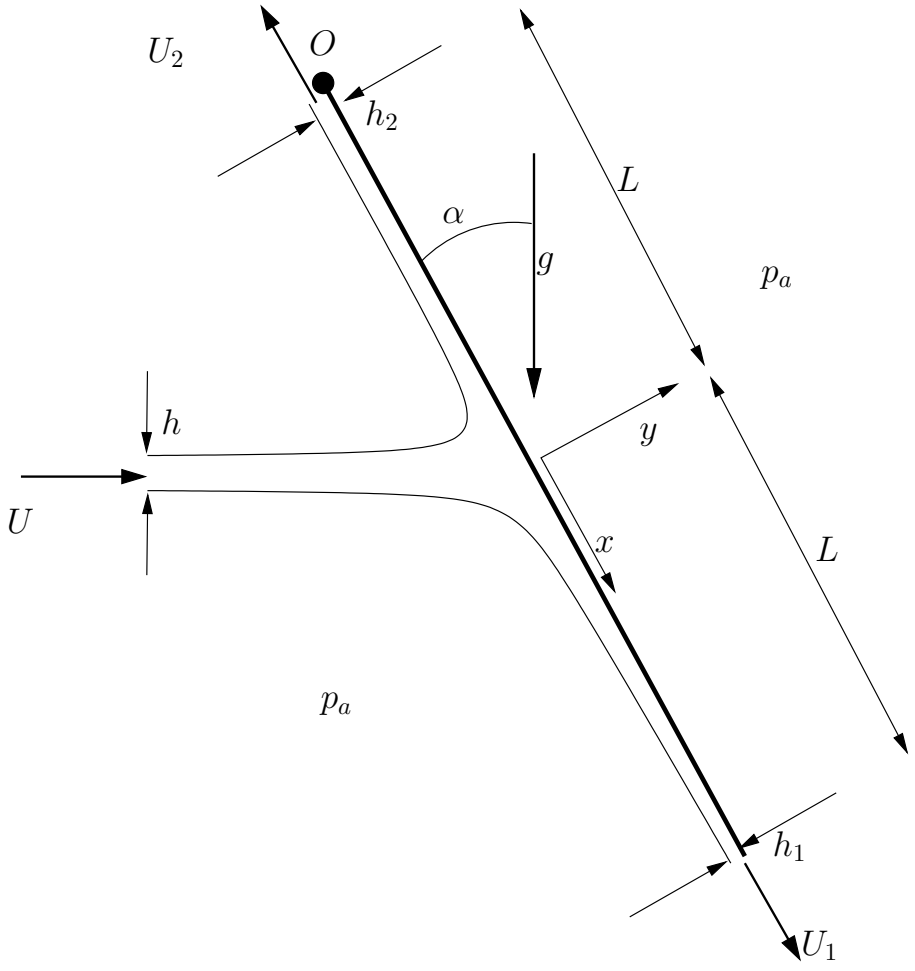
INTRODUCTION TO FLUID MECHANICS. **14-11-2016**
PROBLEM 1 (10 POINTS). **Time: 60 minutes**

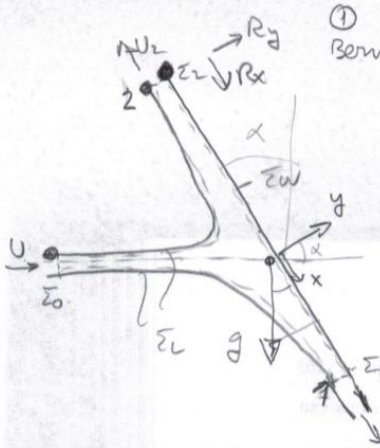
NAME:..... **GROUP:.....**

The sketch below is based on a photograph from an experiment; it shows a jet of liquid impinging on a plate of mass M, per unit length, hinged at point O. In order to charaterize the equilibrium position, in the case of **negligible viscous effects**:

1. Find the velocities U_1 and U_2, at the edges of the plate, where the velocity is parallel to the plate.
2. Find the net force exerted by the fluids on the plate, $\bar{F} = (F_x, F_y)$. Obtain the reaction at the hinge $\bar{R} = (R_x, R_y)$.
3. Write a system of two equations to determine h_1 and h_2.
4. Find the net moment $\bar{M} = M_z \bar{e}_z$, about point O, exerted by the fluids on the plate. Use this result to obtain α.
5. If the heat flux, per unit length, from the liquid to the plate is Q, and the exit temperatures are T_1 and T_2, determine the temperature T at the entrance.

The volume –per unit length– of the control volume sketched in the figure is experimentally known to be V_0 and its centroid is at the origin of the reference system. The thicknesses h, h_1 and h_2 are small compared with L. The whole system is immersed in the atmosphere, at pressure p_a





① Bernoulli: neglect

streamline 1: $P_a + \rho g \frac{h}{2} + \frac{1}{2}\rho U^2 = P_a + \rho g L\cos\alpha + \frac{1}{2}\rho U_1^2$

$$U_1 = \sqrt{U^2 + 2gL\cos\alpha}$$

streamline 2: $P_a + \rho g \frac{h}{2} + \frac{1}{2}\rho U^2 = P_a + \rho g L\cos\alpha + \frac{1}{2}\rho U_2^2$

$$U_2 = \sqrt{U^2 - 2gL\cos\alpha}$$

② Momentum: $\frac{d}{dt}\int_{V_c}\rho\bar{v}\,dV + \int_{\Sigma_c}\rho\bar{v}(\bar{v}-\bar{v}_c)\cdot\bar{n}\,d\sigma = \int_{\Sigma_c}-p\bar{n}\,d\sigma + \int_{\Sigma_c}\bar{\tau}\cdot\bar{n}\,d\sigma + \int_{V_c}\rho\bar{f}_m\,dV$

$$-\rho U h \begin{Bmatrix} U\sin\alpha \\ U\cos\alpha \end{Bmatrix} + \rho U_1 h_1 \begin{Bmatrix} U_1 \\ 0 \end{Bmatrix} + \rho U_2 h_2 \begin{Bmatrix} -U_2 \\ 0 \end{Bmatrix} = -\begin{Bmatrix} F_x \\ F_y \end{Bmatrix} - \rho g V_0 \begin{Bmatrix} -\cos\alpha \\ \sin\alpha \end{Bmatrix}$$

Force exerted by fluids.

Notice $F_x = 0$ (No viscousity)

$$F_y = \rho U^2 h\cos\alpha - \rho g V_0 \sin\alpha$$

PLATE EQUILIBRIUM:

$F_x + Mg\cos\alpha + R_x = 0 \rightarrow \boxed{R_x = -Mg\cos\alpha}$

$F_y - Mg\sin\alpha + R_y = 0 \rightarrow \boxed{R_y = -F_y + Mg\sin\alpha}$

③ Continuity + x-momentum:

$$-\rho U h + \rho U_1 h_1 + \rho U_2 h_2 = 0$$

$$-\rho U^2 h\sin\alpha + \rho U_1^2 h_1 - \rho U_2^2 h_2 = \rho g V_0\cos\alpha$$

$$\left.\begin{aligned} U_1 h_1 + U_2 h_2 &= Uh \\ U_1^2 h_1 - U_2^2 h_2 &= U^2 h\sin\alpha + g V_0\cos\alpha \end{aligned}\right\} \Rightarrow h_1, h_2$$

④ Angular momentum about O:

$$\frac{d}{dt}\int_{V_c}(\bar{x}-\bar{x}_0)\wedge\rho\bar{v}\,dV + \int_{\Sigma_c}(\bar{x}-\bar{x}_0)\wedge\rho\bar{v}(\bar{v}-\bar{v}_c)\cdot\bar{n}\,d\sigma = \int_{\Sigma_c}(\bar{x}-\bar{x}_0)\wedge(-p\bar{n})\,d\sigma + \int_{\Sigma_c}(\bar{x}-\bar{x}_0)\wedge\bar{\tau}\cdot\bar{n}\,d\sigma + \int_{V_c}(\bar{x}-\bar{x}_0)\wedge\rho\bar{g}\,dV$$

$$-\rho U^2 h L\cos\alpha\,\bar{e}_z + \rho U_2^2 h_2 \frac{h_2}{2}\bar{e}_z - \rho U_1^2 h_1 \frac{h_1}{2}\bar{e}_z = -\bar{M} + \rho V_0 g L\sin\alpha\,\bar{e}_z$$

$h_2 \ll L$ NEGLECT. $h_1 \ll L$

$\bar{M}$: MOMENT EXERTED BY FLUIDS

$$\bar{M} = +\rho U^2 h L\cos\alpha\,\bar{e}_z + \rho g V_0 L\sin\alpha\,\bar{e}_z$$

PLATE EQUILIBRIUM: $\sum\bar{M} = 0$:

$$+\rho U^2 h L\cos\alpha + \rho g V_0 L\sin\alpha - Mg L\sin\alpha = 0$$

$$\tan\alpha = \frac{\rho U^2 h L}{MgL - \rho g V_0 L} = \frac{\rho U^2 h}{(M - \rho V_0)g}$$

⑤ ENERGY:

$$\frac{d}{dt}\int_{V_c}\rho e\,dV + \int_{\Sigma_c}\rho e(\bar{v}-\bar{v}_c)\cdot\bar{n}\,d\sigma = \int_{\Sigma_c}-\bar{q}\cdot\bar{n}\,d\sigma + \int_{V_c}(\text{Ch} + C_H)\,dV + \int_{V_c}(-p\bar{\alpha}\cdot\bar{v})\,dV + \int_{V_c}\phi\,dV$$

$$-\rho U h c T + \rho U_1 h_1 c T_1 + \rho U_2 h_2 c T_2 = -Q$$

$$\boxed{T = \frac{U_1 h_1 T_1 + U_2 h_2 T_2}{Uh} - \frac{Q}{\rho U h c}}$$

ESCUELA POLITÉCNICA SUPERIOR

INTRODUCTION TO FLUID MECHANICS. **14-11-2016**
PROBLEM 2 (10 POINTS). **Time: 60 minutes**

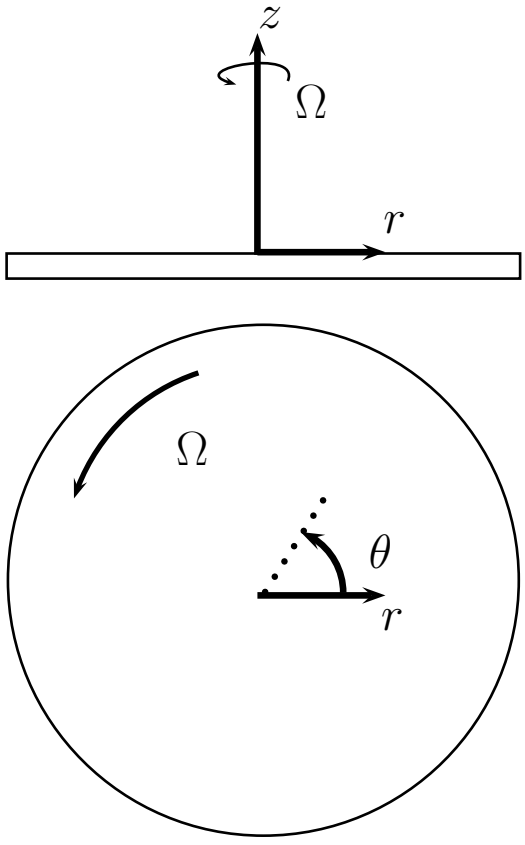
NAME:..... **GROUP:.....**

We will consider an infinitely large porous disk rotating with angular velocity Ω that sucks the fluid with a known velocity $v_z = -2(\nu\Omega)^{1/2}\lambda_1$. The velocity induced in the fluid of density ρ and kinematic viscosity $\nu = \mu/\rho$ is three-dimensional, with components (v_r, v_θ, v_z) in the cylindrical coordinates system shown in the sketch. omega

We want to show that the solution of the problem has the form $v_z = -2(\nu\omega)^{1/2}f(\eta)$, $v_\theta = r\Omega g(\eta)$, where $\eta = (\Omega/\nu)^{1/2}z$ is a similarity variable and $f(\eta)$ and $g(\eta)$ are two functions that must be obtained in the course of the solution, such that the solution far from the disk, as $\eta \to \infty$, is given by $v_r \to 0$, $v_\theta \to 0$.

Taking into account the azimuthal symmetry of the problem,

1. Use the continuity equation to obtain the radial component v_r in terms of the function $f(\eta)$.
2. Use the azimuthal momentum equation to find the ordinary differential equation to be satisfied by the function $g(\eta)$.
3. Use the physical boundary conditions for $\bar{v}$ to find the boundary conditions that $f(\eta)$ and $g(\eta)$ must satisfy.
4. Use the vertical momentum equation to obtain the pressure distribution in terms of an integration function $C(r)$ that will appear in the course of the integration.
5. Use the radial momentum equation to determine $C(r)$ and the ordinary differential equation to be satisfied by $f(\eta)$, with its corresponding boundary conditions. Use the variable separation method to show that $C(r) = A$, where A is a constant. Obtain A in terms of the pressure far from the plate p_∞ and the pressure distribution $p(\eta)$.



$v_z = -2(\nu\Omega)^{1/2} f(\eta)$ where $\eta = \left(\frac{\Omega}{\nu}\right)^{1/2} \cdot z$

$v_\theta = r\Omega\, g(\eta)$

① CONTINUITY: $\frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \frac{\partial v_z}{\partial z} = 0 \Rightarrow \boxed{v_r = r\Omega f'(\eta)}$

② θ-MOMENTUM: $\rho v_r \frac{\partial v_\theta}{\partial r} + \frac{\rho v_r v_\theta}{r} + \rho v_z \frac{\partial v_\theta}{\partial z} = \mu \frac{\partial^2 v_\theta}{\partial z^2}$

$\rho r\Omega^2 g + \rho r\Omega^2 f' g - 2(\nu\Omega)^{1/2} f \cdot \left(\frac{\Omega}{\nu}\right)^{1/2} g' = \mu r\Omega g'' \frac{\Omega}{\nu}$

$\boxed{g'' + 2(fg' - f'g) = 0}$

③ B.C: $z=0$: $v_r = 0 = r\Omega f'(0)$

$v_\theta = \Omega r = r\Omega g(0)$

$v_z = -2(\nu\Omega)^{1/2}\lambda_1 = -2(\nu\Omega)^{1/2} f(0)$

$z \to \infty$: $v_r = 0 \Rightarrow f'(\infty) = 0$

$v_\theta = 0 \Rightarrow g(\infty) = 0$

$\left.\begin{array}{l} f(0) - \lambda_1 = f'(0) = 0 \\ f'(\infty) = 0 \\ g(0) = 1 \\ g(\infty) = 0 \end{array}\right\}$

④ Z-MOMENTUM: $\rho v_z \frac{\partial v_z}{\partial z} = -\frac{\partial p}{\partial z} + \mu \frac{\partial^2 v_z}{\partial z^2} \rightarrow \frac{\partial}{\partial z}\left[p + \frac{1}{2}\rho v_z^2 - \mu \frac{\partial v_z}{\partial z}\right] = 0$

$p + \frac{1}{2}\rho v_z^2 - \mu \frac{\partial v_z}{\partial z} = C(r) \rightarrow \boxed{p = C(r) - 2\mu\Omega\left[f^2 + f'\right]}$

⑤ r-MOMENTUM: $\rho v_r \frac{\partial v_r}{\partial r} - \frac{\rho v_\theta^2}{r} + \rho v_z \frac{\partial v_r}{\partial z} = -\frac{\partial p}{\partial r} + \mu \frac{\partial^2 v_r}{\partial z^2}$

$\rho r\Omega^2 f'^2 - \rho r\Omega^2 g^2 - 2(\nu\Omega)^{1/2} f \cdot r\Omega \left(\frac{\Omega}{\nu}\right)^{1/2} \cdot f'' = -C' + \mu r\Omega f''' \cdot \frac{\Omega}{\nu}$

$-f''' + f'^2 - g^2 - 2ff'' = -\frac{C'(r)}{\rho r\Omega^2} = \text{CONST} = -\lambda$

$f''' + f'^2 - g^2 - 2ff''$: FUNCTION OF η; $\frac{C'(r)}{\rho r\Omega^2}$: FUNCTION OF r; $\uparrow$ NECESSARILY

as $\eta \to \infty$: $f' \to 0 \Rightarrow f'' \to 0 \Rightarrow f''' \to 0$, $g \to 0$ $\Rightarrow \lambda = 0 \Rightarrow C'(r) = 0 \Rightarrow \boxed{C(r) = A}$

as $\eta \to \infty$: $p = p_\infty = A - 2\mu\Omega\left[f^2(\infty) - f'(\infty)\right] \Rightarrow A = p_\infty + 2\mu\Omega f^2(\infty)$

$\boxed{p = p_\infty - 2\mu\Omega\left[f^2(\eta) - f^2(\infty) + f'(\eta)\right]}$