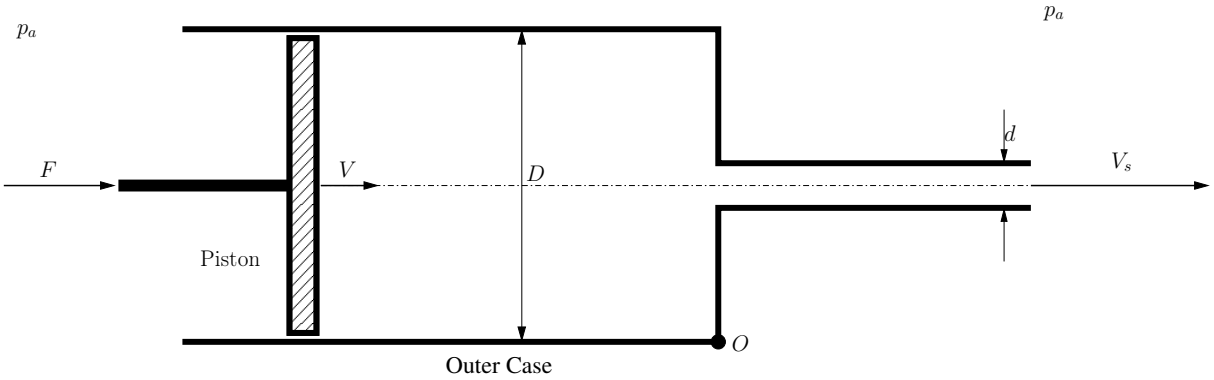
ESCUELA POLITÉCNICA SUPERIOR

INTRODUCTION TO FLUID MECHANICS. **05-11-2014**
PROBLEM 1 (10 POINTS). **Time: 60 minutes**

NAME:..... **GROUP:.....**

The figure shows an injector made of an outer case and a piston od diameter D moving with a velocity V relative to the case. The system is designed to be supported by a single point O and the piston will be moved, without contact with the outer case, by an axial actuator exerting a force F. At $t = 0$, the piston is located at a distance L from the contraction, and the system is full of liquid of density ρ, viscosity μ, heat capacity c and conductivity k (all of them constant).



1. Determine the discharge velocity V_s in the atmosphere, as a function of the data.
2. Determine the force exerted by the fluid on both piston and outer case, $\bar{F}_P + \bar{F}_C$.
3. Determine the moment exerted by the fluid on both piston and outer case $\bar{M}_{O,P} + \bar{M}_{O,C}$, about point O.

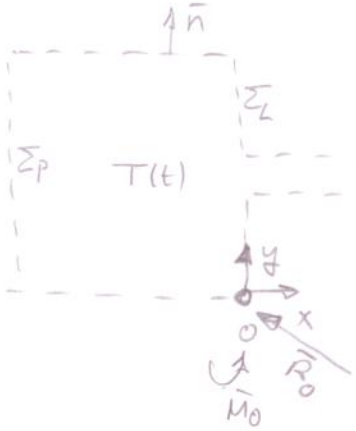
For **questions 2 and 3** consider the unsteady term to be negligible, because $d/D \ll 1$

4. Using Bernoulli equiation, find the pressure on the piston surface p_P, assuming this pressure to be uniform. Indicate the applicability of the result.
5. Taking into account the last result, find the separate results for forces exerted by the fluid on the piston and on the case. Find the force exerted by the actuator and reaction on point O (only the force).

If the liquid is initially at a temperature T_0 and the process is slow enough so that we can consider the liquid temperature to be uniform,

6. Find the evolution of the liquid temperature as a function of time $T(t)$.

Consider that the net heat transfer to the liquid from the walls (piston and case) is a constant $\dot{Q}$ and neglect viscous dissipation.



① CONTINUITY

$$\frac{d}{dt}\int_{V_c} \rho\, dV + \int_{\Sigma_c} \rho(\bar{v}-\bar{v}_c)\cdot\bar{n}\, d\sigma = 0$$

Σ_P: $\bar{v}=\bar{v}_c$; Σ_L: $\bar{v}=\bar{v}_c=0$; Σ_e

$$V(t) = (L - Vt)\frac{\pi D^2}{4} \rightarrow \frac{dV}{dt} = -V\cdot\frac{\pi D^2}{4}$$

$$-\rho\frac{\pi D^2}{4}V + \rho\frac{\pi d^2}{4}V_s = 0 \Rightarrow \boxed{V_s = V\left(\frac{D}{d}\right)^2}$$

② MOMENTUM:

$$\frac{d}{dt}\int_{V_c}\rho\bar{v}\,dV + \int_{\Sigma_c}\rho\bar{v}(\bar{v}-\bar{v}_c)\cdot\bar{n}\,d\sigma = \int_{\Sigma_c} -p\bar{n}\,d\sigma + \int_{\Sigma_c}\bar{\tau}'\cdot\bar{n}\,d\sigma + \int_{V_c}\rho\bar{g}\,dV$$

NEGLECT. Σ_P, Σ_c, Σ_e; $\bar{\tau}'$: Σ_P, Σ_c, Σ_e; $\bar{g}$: UNIFORM

$$= -\bar{F}_{L\to P} - \bar{F}_{L\to C}$$

$$\frac{\text{UNSTEADY}}{\text{CONVECTIVE}} \sim \frac{\rho V^2\frac{\pi D^2}{4}}{\rho V_s^2\frac{\pi d^2}{4}} = \left(\frac{d}{D}\right)^2 \ll 1$$

$$\bar{F}_{L\to P} + \bar{F}_{L\to C} = -\rho V_s^2\frac{\pi d^2}{4}\bar{e}_x - p_a\frac{\pi d^2}{4}\bar{e}_x$$

$$\bar{F}_{A\to P} + \bar{F}_{A\to C} = \underbrace{-\int_{\Sigma_P+\Sigma_C} p_a\bar{n}\,d\sigma}_{=0} + \underbrace{\int_{\Sigma_e} p_a\bar{n}\,d\sigma}_{=0} + \int_{\Sigma_e} p_a\bar{n}\,d\sigma = p_a\frac{\pi d^2}{4}\bar{e}_x$$

$$\boxed{\bar{F}_P + \bar{F}_C = -\rho V_s^2\frac{\pi d^2}{4}\bar{e}_x}$$

③ ANGULAR MOMENTUM:

$$\frac{d}{dt}\int_{V_c}\bar{x}_A\times\rho\bar{v}\,dV + \int_{\Sigma_c}\bar{x}_A\times(\rho\bar{v}(\bar{v}-\bar{v}_c)\cdot\bar{n})\,d\sigma = \int_{\Sigma_c} -\bar{x}_A\times p\bar{n}\,d\sigma + \int_{\Sigma_c}\bar{x}_A\times(\bar{\tau}'\cdot\bar{n})\,d\sigma + \int_{V_c}\bar{x}_A\times\rho\bar{g}\,dV$$

Σ_P, Σ_c, Σ_e; $\bar{\tau}'$: Σ_P, Σ_c, Σ_e; $\bar{g}$: V_c

$$= -\bar{M}_{L\to C} - \bar{M}_{L\to P}$$

$$\bar{M}_{L\to P} + \bar{M}_{L\to C} = \rho V_s^2\pi\frac{d^2}{4}\frac{D}{2}\bar{e}_z + p_a\cdot\pi\frac{d^2}{4}\cdot\frac{D}{2}\bar{e}_z$$

$$\bar{M}_{A\to P} + \bar{M}_{P\to C} = \underbrace{-\int_{\Sigma_P+\Sigma_C}\bar{x}_A\times p_a\bar{n}\,d\sigma}_{=0} - \underbrace{\int_{\Sigma_e}\bar{x}_A\times p_a\bar{n}\,d\sigma}_{=0} + \int_{\Sigma_e}\bar{x}_A\times p_a\bar{n}\,d\sigma$$

$$\boxed{\bar{M}_P + \bar{M}_C = \rho V_s^2\frac{\pi d^2}{4}\cdot\frac{D}{2}\bar{e}_z}$$

④ Bernoulli, valid if viscosity negligible, $\frac{\partial}{\partial t} = 0$, ρ = const.

$$p_P + \frac{1}{2}\rho V^2 = p_a + \frac{1}{2}\rho V_s^2 \Rightarrow \boxed{p_P = p_a + \frac{1}{2}\rho V^2\left[\left(\frac{D}{d}\right)^4 - 1\right]}$$

⑤ $\bar{F}_p = -(p_p - p_a)\frac{\pi D^2}{4}\bar{e}_x = -\frac{1}{2}\rho V^2\left[\left(\frac{D}{d}\right)^4 - 1\right]\frac{\pi D^2}{4}\bar{e}_x$

$\bar{F}_p + \bar{F}_c = -\rho v_s^2 \frac{\pi d^2}{4}\bar{e}_x$

$\bar{F}_c = -\rho v_s^2 \frac{\pi d^2}{4} + \frac{1}{2}\rho V^2\left[\left(\frac{D}{d}\right)^4 - 1\right]\frac{\pi D^4}{4} = \frac{1}{2}\rho V^2 \frac{\pi D^2}{4}\left[\left(\frac{D}{d}\right)^4 - 1 - 2\left(\frac{D}{d}\right)^2\right]\bar{e}_x$

$\bar{M}_p = (p_p - p_a)\frac{\pi D^2}{4}\cdot\frac{D}{2}\bar{e}_z = \frac{1}{2}\rho V^2 \frac{\pi D^2}{4}\cdot\frac{D}{2}\left[\left(\frac{D}{d}\right)^4 - 1\right]\bar{e}_z$

$\bar{M}_c = \frac{2}{2}\rho V^2 \frac{\pi d^2}{4}\frac{D}{2}\left[\left(\frac{D}{d}\right)^4 - 1\right]\bar{e}_z - \frac{1}{2}\rho V^2 \frac{\pi D^2}{4}\frac{D}{2}\left[\left(\frac{D}{d}\right)^4 - 1\right]\bar{e}_z$

$\bar{M}_c = -\frac{1}{2}\rho V^2 \frac{\pi D^2}{4}\cdot\frac{D}{2}\cdot\left[\left(\frac{D}{d}\right)^4 - 1 - 2\left(\frac{D}{d}\right)^2\right]\bar{e}_z$

PISTON MOMENTUM: $M\frac{d\bar{v}}{dt} = F + F_p \Rightarrow F = -F_p = \frac{1}{2}\rho V^2\left[\left(\frac{D}{d}\right)^4 - 1\right]\frac{\pi D^2}{4}$ (the $M\frac{d\bar{v}}{dt}$ term $= 0$)

CASE MOMENTUM: $\bar{F}_c + \bar{R}_0 = 0 \Rightarrow \bar{R}_0 = -\bar{F}_c = -\frac{1}{2}\rho V^2 \frac{\pi D^2}{4}\left[\left(\frac{D}{d}\right)^4 - 1 - 2\left(\frac{D}{d}\right)^2\right]\bar{e}_x$

CASE ANGULAR MOMENTUM: $\bar{M}_c + \bar{M}_0 = 0 \Rightarrow \bar{M}_0 = -\bar{M}_c$

⑥ ENERGY:

$\frac{d}{dt}\int_{V_c}\rho e\,dV + \int_{\Sigma_c}\rho e(\bar{v} - \bar{v}_c)\cdot\bar{n}\,d\Sigma = \underbrace{\int_{\Sigma_c} -\bar{q}\cdot\bar{n}\,d\Sigma}_{\dot{Q}} + \int_{V_c}(Q_c + Q_r)\,dV + \int_{V_c} -p\,\bar{v}\cdot\bar{v}\,dV + \int_{V_c}\phi\,dV$

$\frac{d}{dt}\left[\rho c T V\right] + \rho c T v_s \frac{\pi d^2}{4} = \dot{Q}$

$\rho c\left[T\left(\frac{dV}{dt} + v_s\frac{\pi d^2}{4}\right) + V\frac{dT}{dt}\right] = \dot{Q}$

($\frac{dV}{dt} + v_s\frac{\pi d^2}{4} = 0$ CONTINUITY)

$\rho c\,(L - vt)\cdot\frac{\pi D^2}{4}\frac{dT}{dt} = \dot{Q}$

$dT = \frac{\dot{Q}\,4\,dt}{\rho c\,\pi D^2\,L - vt}$

$T - T_0 = \frac{4\dot{Q}}{\rho c \pi D^2 V}\ln\left[\frac{L}{L - vt}\right]$

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PROBLEM 2 (10 POINTS). **Time: 60 minutes**

NAME:..... **GROUP:.....**

The radius of a sphere immersed in a liquid evolves in time acccording to $R(t) = R_0 + a\sin(\omega t)$. The liquid is initially at rest and, far from the sphere, the pressure remains constant p_∞.

1. Comment the expected features of the velocity and pressure fields that allow simplification of the problem. Would it be applicable for any set of initial conditions initial conditions?
2. Find the velocity field $\vec{v}(r,t)$
3. Find the pressure field $p(r,t)$
4. Cavitation is a phenomenon that occurs when pressure in the gas decreases below the vapor pressure of the liquid p_v at some point; In such case, the liquid will vaporize. Find the critical condition for cavitation to occur at the surface of the sphere if $R_0 > 5a$. Take $p_v \simeq 0$ (p_v is small compared with typical values of the pressure).
5. Evaluate the instaneous amount of energy dissipated per unit time in the whole fluid domain and the total amount of energy dissipated in a cycle in the whole fluid domain.

① IF B.C. + I.C. SPHERICALLY SYMMETRICAL => SOLUTION SPHERICALLY SYMMETRICAL

$$p = p(r,t)$$
$$\bar{v} = \bar{v}(r,t) = v_r(r,t)\,\bar{e}_r$$

IF INITIAL CONDITIONS NOT SPHERICALLY SYMMETRICAL =>

=> IN GENERAL $p = p(r,\theta,\phi,t)$ NOT SPHERICALLY SYMMETRICAL.

$$\bar{v} = \bar{v}(r,\theta,\phi,t)$$

② CONTINUITY: $\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) = 0 \Rightarrow r^2 v = F(t)$

$r = R(t) : v = \dot{R}(t) = a\omega\cos\omega t$

$$\boxed{v(r) = \frac{a\omega\cos\omega t\,[R_0 + a\sin\omega t]^2}{r^2}}$$

③ MOMENTUM-r: $\rho\left[\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial r}\right] = -\frac{\partial p}{\partial r} + \mu\frac{\partial}{\partial r}\left[\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v)\right] \rightarrow \frac{\partial}{\partial r}\left(p + \frac{1}{2}\rho v^2\right) = -\rho\frac{\partial v}{\partial t}$

$$\frac{\partial}{\partial r}\left(p + \frac{1}{2}\rho v^2\right) = \frac{\rho a\omega^2\sin\omega t\,[R_0 + a\sin\omega t]^2 - 2\rho a^2\omega^2\cos^2\omega t\,[R_0 + a\sin\omega t]}{r^2}$$

$$\boxed{p = p_\infty - \frac{1}{2}\rho a^2\omega^2\cos^2\omega t\,\frac{[R_0 + a\sin\omega t]^4}{r^4} - \frac{\rho a\omega^2\sin\omega t\,[R_0 + a\sin\omega t]^2}{r} + \frac{2\rho a^2\omega^2\cos^2\omega t\,[R_0 + a\sin\omega t]}{r}}$$

④ $p(R) = p_\infty - \frac{1}{2}\rho a^2\omega^2\cos^2\omega t - \rho a\omega^2\sin\omega t\,[R_0 + \sin\omega t] + 2\rho a^2\omega^2\cos^2\omega t$

$$= p_\infty + \rho a\omega^2\left[-\sin\omega t\,(R_0 + a\sin\omega t) + \frac{3}{2}\rho a^2\cos^2\omega t\right]$$

$$= p_\infty + \rho a\omega^2\left[-\sin\omega t\,(R_0 + a\sin\omega t) - a\sin\omega t\cos\omega t - 3\rho a\cos\omega t\sin\omega t\right]$$

EXTREMA: $\frac{dp(R)}{dt} = \rho a\omega^3\left[-\cos\omega t\,(R_0 + a\sin\omega t) - a\sin\omega t\cos\omega t - 3\rho a\cos\omega t\sin\omega t\right]$

$$= \rho a\omega^3\left[-\cos\omega t\,(R_0 + 5a\sin\omega t)\right]$$

$\frac{dp(R)}{dt} = 0 \rightarrow \cos\omega t = 0 : \begin{cases} \omega t = \frac{\pi}{2} + 2n\pi \\ \omega t = \frac{3\pi}{2} + 2n\pi \end{cases} \quad n = 0, 1, 2, \ldots$

$\rightarrow R_0 + 5a\sin\omega t = 0$ NO, IF $R_0 > 5a$

$$\frac{d^2p(R)}{dt^2} = \rho a\omega^4\left[\sin\omega t\,(R_0 + 5a\sin\omega t) - 5a\cos^2\omega t\right]$$

> 0 $\omega t = \frac{\pi}{2} + 2n\pi$ MINIMUM.

< 0 $\omega t = \frac{3\pi}{2} + 2n\pi$ MAXIMUM.

$$\boxed{p(R)_{MINIMUM} = p_\infty - \rho a\omega^2(R_0 + a) < 0 \quad \text{CAVITATION}}$$

⑤ $\phi_v = 2\mu\left[\left(\frac{\partial v}{\partial r}\right)^2 + \left(\frac{v}{r}\right)^2\right]$; $\frac{\partial v}{\partial r} = -\frac{2a\omega\cos\omega t\,R^2}{r^3}$; $\left(\frac{\partial v}{\partial r}\right)^2 = \frac{4a^2\omega^2\cos^2\omega t\,R^4}{r^6}$; $\left(\frac{v}{r}\right)^2 = \frac{a^2\omega^2\cos^2\omega t\,R^4}{r^6}$

$$\phi_v = \frac{10\mu a^2\omega^2\cos^2\omega t\,R^4}{r^6}$$

$$\dot{W}|_{CYCLE} = \int_R^\infty \phi_v \cdot 4\pi r^2 dr = \frac{40\pi}{3}\mu a^2\omega^2\cos^2\omega t \cdot R$$

$$E_{CYCLE} = \int_0^T \dot{W}\,dt = \frac{40\pi\mu}{3}a^2\omega^2\underbrace{\int_0^T \cos^2\omega t \cdot R(t)\,dt}_{R_0\cdot\frac{T}{2}} = \frac{40\pi^2\mu}{3}a^2\omega R_0 \qquad T = \frac{2\pi}{\omega}$$