FLUID MECHANICS I **19-01-2022**
THEORY (5 POINTS). **Time: 30 minutes**

NAME:..... **GRADE:.....**

We will consider the analysis of fluid dynamics in a cartesian reference system rotating with constant angular velocity $\bar{\Omega} = \Omega \bar{e}_z$. Notice this is a **non-inertial** reference system and inertial forces need to be taken into account, in particular the centrifugal force $-\bar{\Omega} \wedge (\bar{\Omega} \wedge \bar{x})$ and the Coriolis force $-2\bar{\Omega} \wedge \bar{v}$, where $\bar{x} = (x, y, z)$ is the position of a generic point in this system of reference, defined by the cartesian unit vectors $(\bar{e}_x, \bar{e}_y, \bar{e}_z)$, and $\bar{v}(x, y, z)$ is the velocity vector of the fluid at that point, relative to the above defined reference system.

1. Show that the centrifugal force derives from a force potential U such that $\bar{f}_m = -\nabla U$ and determine U.

2. Determine the contribution of the Coriolis force to the energy equation

3. Derive the Bernoulli equation for steady incompressible flows with negligible viscous effects, in the above defined reference system, if the gravitational force per unit mass is $-g\bar{e}_z$. Use the momentum equation and project in the direction of the streamline $\bar{v}/v$

Note:
Continuity equation: $\partial\rho/\partial t + \nabla \cdot (\rho \bar{v}) = 0$
Momentum equation: $\rho \partial \bar{v}/\partial t + \rho \nabla (v^2/2) - \rho \bar{v} \wedge (\nabla \wedge \bar{v}) = -\nabla p + \nabla \cdot \tau' - \rho g \bar{e}_z - \rho \bar{\Omega} \wedge (\bar{\Omega} \wedge \bar{x}) - 2\rho \bar{\Omega} \wedge \bar{v}$

① $\left.\begin{aligned} -\frac{\partial U}{\partial x} &= \Omega^2 x \\ -\frac{\partial U}{\partial y} &= \Omega^2 y \\ -\frac{\partial U}{\partial z} &= 0 \end{aligned}\right\} \Rightarrow U = U(x,y) = -\frac{\Omega^2(x^2+y^2)}{2} + C$

NOTICE THAT EXISTENCE CAN BE PROVED BY CHECKING $\nabla \times \bar{F}_{cent} = 0$

② VOLUMETRIC FORCE IN ENERGY → $\rho \bar{f}_m \cdot \bar{v} \overset{\text{CORIOLIS}}{\Rightarrow} -2\rho(\bar{\Omega} \wedge \bar{v}) \cdot \bar{v} \equiv 0$

③ MOMENTUM (STEADY, INVISCID, INCOMPRESSIBLE):

$$\nabla\left(\frac{v^2}{2}\right) - \bar{v} \wedge (\nabla \wedge \bar{v}) = -\nabla\left(\frac{p}{\rho}\right) - g\bar{e}_z - \rho\bar{\Omega} \wedge (\bar{\Omega} \wedge \bar{x}) - 2\rho\bar{\Omega} \wedge \bar{x}$$

PROJECT $\cdot \frac{\bar{v}}{|\bar{v}|}$ $\Rightarrow$ $\frac{\partial}{\partial l}\left(\frac{v^2}{2} + \frac{p}{\rho}\right) = -g\bar{e}_z \cdot \frac{\bar{v}}{|\bar{v}|} - \rho\left[\bar{\Omega} \wedge (\bar{\Omega} \wedge \bar{x})\right] \cdot \frac{\bar{v}}{|\bar{v}|} + 0$

POTENTIAL (GRAVITY + CENTRIFUGAL): $-g\bar{e}_z - \rho\bar{\Omega} \wedge (\bar{\Omega} \wedge \bar{x}) = -\nabla\left(gz + \frac{\Omega^2(x^2+y^2)}{2}\right)$

SO WE GET

$$\frac{\partial}{\partial l}\left(p + \frac{1}{2}\rho v^2 + \rho g z - \rho\frac{\Omega^2(x^2+y^2)}{2}\right) = 0$$

$$\boxed{p + \frac{1}{2}\rho v^2 + \rho g z - \rho\frac{\Omega^2(x^2+y^2)}{2} = C_l}$$

FLUID MECHANICS I **19-01-2022**
PROBLEM 1 (10 POINTS). **Time: 40 minutes**

NAME:..... **GRADE:.....**

The sketch shown in figure 1 depicts a micropumping device. The pipe of radius a has a vertical arm of length H and a horizontal arm of length $R \sim H \gg a$, and rotates with a **constant** angular velocity $\Omega\bar{e}_z$, and takes liquid from a low level reservoir. In the anaysis, a reference system attached to the rotating pipe, shown in the sketch, is suggested (notice this is a rotating system, i.e. non inertial), where the reduced manometric pressure P can be written as $P = p - p_a + \rho g z - \rho\Omega^2(x^2+y^2)/2$, in terms of the vertical distance z and the radial distance to the rotation axis $r = (x^2+y^2)^{1/2}$, to include both the gravity force and the centrifugal force. The Coriolis force can be neglected in the flow along the pipe. Using dimensional analysis, we would like to characerize the flow rate Q and the reduced pressure difference ΔP between the inlet and outlet sections of the pipe in terms of the geometry (a, R, H), velocity (ΩR), gravity (gH) and fluid properties (density ρ and viscosity μ).

1. Make use the Π-theorem to reduce the parametric depence of the non-dimensional flow rate parameter Q and reduced pressure difference ΔP. Use the tube radius a as characteristic length.

$$Q = Q(\rho, \mu, gH, \Omega R, H, R, a), \quad \Delta P = \Delta P(\rho, \mu, gH, \Omega R, H, R, a)$$

2. Using the results derived from the non-dimensional analysis, write the continuity and momentum differential [1] equations plus boundary contions in non-dimensional form. To do so, use the scaled variables $\vec{u} = \vec{v}/\Omega R$, $\pi = P/(\rho\Omega^2 R^2)$ and $\vec{r} = \vec{x}/a$. To write the boundary conditions at the entrance of the pipe $z = -H$, consider $|\vec{v}|^2 = -2gH + \Omega^2 R^2$.

3. Give the condition for the viscosity to be negligible and simplify the equation.

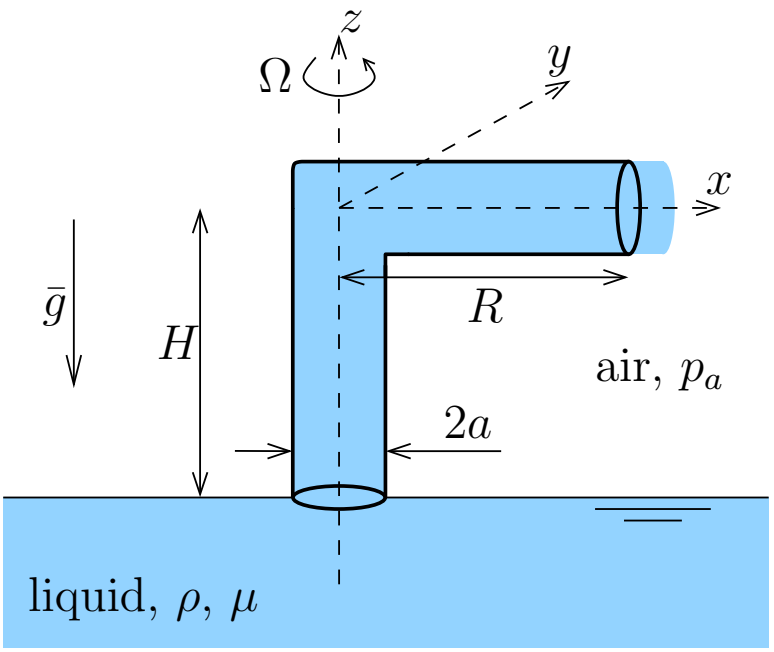
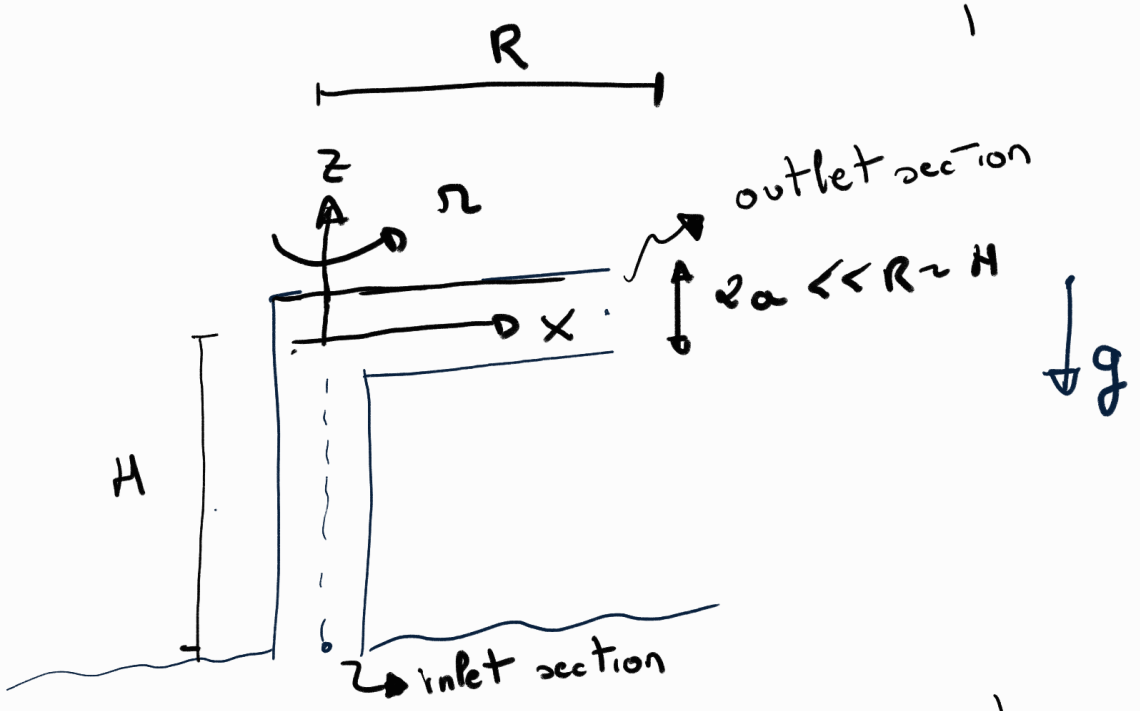


Figura 1: Sketch of the problem

[1] Use equation 4.7 and the steady version of equation 4.9 of the class notes.



1) $Q = Q(\rho, \mu, gH, \Omega R, H, R, a)$

we choose $\rho, \Omega R, a$ as variables with independent dimensions.

$$\frac{Q}{\Omega R \cdot a^2} = Q\left(\frac{\rho \Omega R a}{\mu}, \frac{gH}{\Omega^2 R^2}, \frac{H}{a}, \frac{R}{a}\right)$$

Doing the same with ΔP

$$\frac{\Delta P}{\rho \Omega^2 a^2} = \Delta P\left(\frac{\rho \Omega R a}{\mu}, \frac{gH}{\Omega^2 R^2}, \frac{H}{a}, \frac{R}{a}\right)$$

2) $\bar{u} = \frac{\bar{v}}{\Omega R}$, $\pi = \frac{P}{\rho \Omega^2 R^2}$, $\vec{x}^* = \frac{\vec{x}}{a}$

(x^*, y^*, z^*)

where $\bar{x} = (x, y, z)$ and $\vec{x}^* = \left(\frac{x}{a}, \frac{y}{a}, \frac{z}{a}\right)$

We define $\nabla^* = a \cdot \nabla(\,)$ where $\nabla(\,)$ is the differential operator. Introducing these non-dimensional variables in (2.9)

$$\boxed{\rho \cdot \bar{v} \cdot \nabla \bar{v} = -\nabla p + \mu \nabla^2 \bar{v} + \rho \cdot \bar{f}_m} \quad (2.9)$$

$\vec{f}_m = \bar{g} - \bar{\Omega} \wedge \bar{\Omega} \wedge \bar{x} - \cancel{2\rho \bar{\Omega} \wedge \bar{v}}$ Negligible

Since $\bar{f}_m = -\nabla U$, with $U = -\bar{g} \cdot \bar{x} - \frac{1}{2}(\Omega \wedge \bar{x}) \cdot (\Omega \wedge \bar{x})$

$$\frac{\rho \cdot \Omega^2 R^2}{a} \bar{u} \cdot \nabla^* \bar{u} = -\frac{\rho \Omega^2 R^2}{a} \nabla^* \pi + \frac{\mu \cdot \Omega R}{a^2} \nabla^{*2} \bar{u}.$$

$$\boxed{\bar{u} \cdot \nabla^* \bar{u} = -\nabla^* \pi + \frac{1}{Re} \cdot \nabla^{*2} \bar{u}}$$

To write the boundary conditions we take into account that

$$\pi = \frac{p}{\rho \Omega^2 R^2} = \frac{p - p_a}{\rho \Omega^2 R^2} + \frac{gH}{\Omega^2 R^2} \cdot \frac{a}{H} \cdot z^* - \frac{a^2}{R^2} \cdot \frac{x^{*2} + y^{*2}}{2}$$

At the outlet section $x^* = \frac{R}{a}$, $z^* = 0$

$$\pi\left(x^* = \frac{R}{a}\right) = -\frac{1}{2} \quad \left(\text{we used } \frac{a}{R} << 1\right)$$

At the walls $\bar{x}^* = \bar{x}^*_{walls} \Rightarrow \bar{u} = 0$

At the inlet section $z^* = -\frac{H}{a}$

$$|\bar{u}|^2 = -2\frac{gH}{\Omega^2 R^2} + 1$$

3) If we want viscosity to be negligible we compare the order of magnitude of the terms in equation (2.9)

$$\rho \cdot \bar{v} \nabla \bar{v} = -\nabla p + \mu \nabla^2 \bar{v}$$

$$\rho \frac{\Omega^2 R^2}{a} \qquad \frac{\Delta p}{a} \qquad \mu \cdot \frac{\Omega R}{a^2}$$

$$p = p - p_a + \rho g z - \frac{(\bar{\Omega} \wedge \bar{x}) \cdot (\bar{\Omega} \wedge \bar{x})}{2}$$

$$Re = \frac{\rho \Omega^2 R^2 / a}{\mu \cdot \Omega R / a^2} = \frac{\rho \Omega R a}{\mu} >> 1$$

In this case we can obtain the characteristic pressure variation giving

$$\Delta p \sim \rho \cdot \Omega^2 R^2$$

In the limit $Re >> 1$, the non-dimensional equations yield

$$\nabla^* \cdot \bar{u} = 0$$

$$\bar{u} \cdot \nabla^* \bar{u} = -\nabla \pi$$

with boundary conditions:

Outlet section

$z^* = 0, \; x^* = \frac{R}{a} \qquad \pi = -1/2$

Pipe walls

$\bar{x}^* = \bar{x}^*_{walls} \qquad \bar{u} \cdot \bar{n} = 0$

Inlet section

$x = 0, \; z^* = -\frac{H}{a} \qquad |\bar{u}| = \left(1 - \frac{2gH}{\Omega^2 R^2}\right)^{1/2}$

As predicted by the dimensional analysis, the parametric dependence is reduced, in this limit, to

$$\frac{R}{a}, \frac{H}{a} \text{ and } \frac{gH}{\Omega^2 R^2}$$

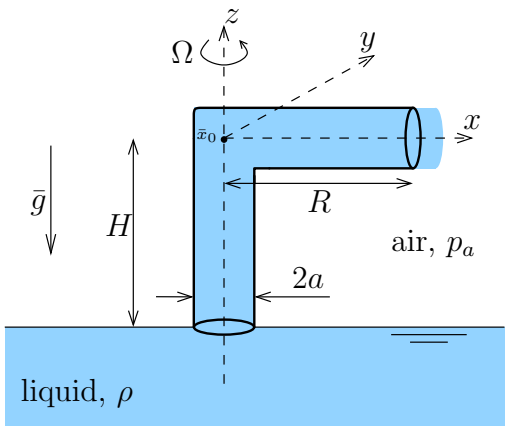
FLUID MECHANICS I **19-01-2022**
PROBLEM 2 (10 POINTS). **Time: 75 minutes**

NAME:..... **GRADE:.....**

The sketch below shows a simple pumping device made with an L-shaped pipe of radius a. The horizontal arm has a length R and the vertical arm has a length H. The lower end of the vertical arm is placed on the free surface of the reservoir from which it takes liquid. The device rotates with a constant angular velocity $\Omega\,\bar{e}_z$.

In this problem, we shall study the regime where **viscous effects are negligible** and we aim to characterize the flow rate (or the mean fluid velocity along the pipe v_m) and the torque in terms of the geometry (a, R, H), the angular velocity (Ω) and the liquid density (ρ).

The use of a reference frame that rotates with $\Omega\,\bar{e}_z$, in which the geometry and the flow are steady, is suggested in this problem. In the rotating reference frame, the inviscid flow along the pipe is purely longitudinal and can be considered uniform in the cross section of the pipe, being $v_\mathrm{m}\,\bar{e}_z$ in the vertical arm and $v_\mathrm{m}\,\bar{e}_x$ in the horizontal arm (v_m to be determined), while the flow in the reservoir will show a solid-body rotation with angular velocity $-\Omega\,\bar{e}_z$.



Recall the expression for the body forces per unit mass in a reference frame rotating with a constant angular velocity $\bar{\Omega}$:

$$\bar{f}_\mathrm{m} = \bar{g} - \bar{\Omega} \wedge (\bar{\Omega} \wedge \bar{x}) - 2\,\bar{\Omega} \wedge \bar{v}.$$

Notice that in the non-inertial reference frame, the potential U can be written as $U = gz - \Omega^2(x^2+y^2)/2$ to include both the gravity force and the centrifugal force in Bernoulli's equation. Bernoulli's theorem can be used since the Coriolis force, although non-conservative, does not do any work on the fluid.

1. Consider the flow along the pipe and find the pressure p_in induced by the rotation at the entrance of the pipe.

2. Consider now the entrance region induced in the reservoir and determine the mean velocity v_m that the liquid acquires in the pipe.

3. Give the condition that Ω has to satisfy for effective pumping.

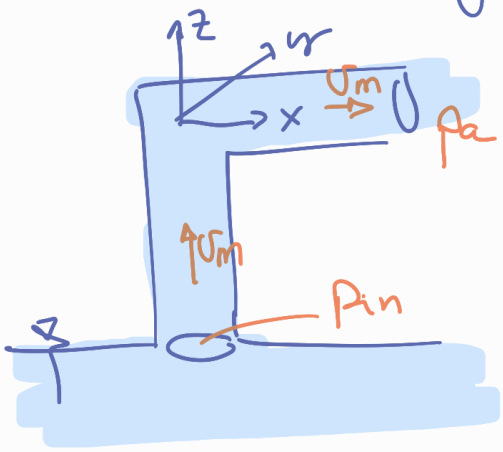
After determination of the minimum value of Ω for effective pumping, we shall find an upper limit by considering the possibility of cavitation. Cavitation is an undesirable phenomena in pumps that takes place when the static pressure decreases to a level equal to the vapor pressure of the liquid p_v and bubbles are formed that reduce the performance of the pump.

4. Determine the point along the pipe where minimum pressure occurs and its value, p_min. Give the condition that Ω has to satisfy to avoid cavitation of the liquid in the device.

Finally, use a control volume analysis to

5. Find the force $\bar{F}$ and the corresponding moment $\bar{M}_0$ (with respect to the point $\bar{x}_0$, in the center of the elbow) that the liquid exerts on the pump. Do not neglect the Coriolis force in this question.

① Bernoulli along the pipe



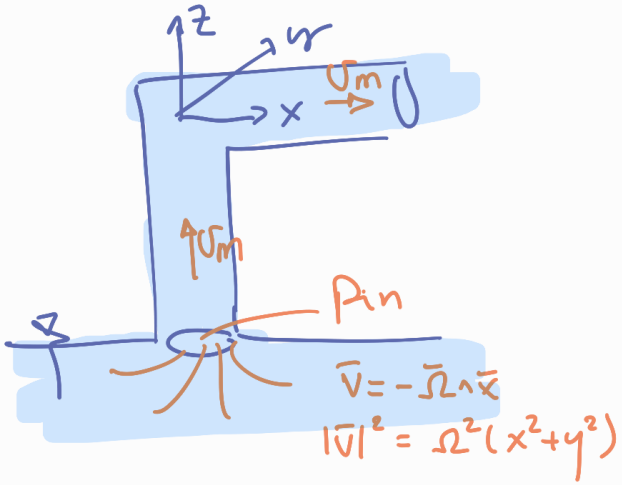
$$p + \frac{1}{2}\rho|\bar{v}|^2 + \rho g z - \rho\frac{\Omega^2(x^2+y^2)}{2} = C$$

$$p_{in} + \frac{1}{2}\rho v_m^2 - \rho g H =$$

$$= p_a + \frac{1}{2}\rho v_m^2 - \rho\frac{\Omega^2 R^2}{2}$$

$$\rightsquigarrow \boxed{p_m - p_a = \rho g H - \rho\frac{\Omega^2 R^2}{2}}$$

② Bernoulli in entrance region in the reservoir



$$p_a + \frac{1}{2}\rho\Omega^2(x^2+y^2) - \rho g H - \rho\frac{\Omega(x^2+y^2)}{2} = p_2 + \frac{1}{2}\rho v_m^2 - \rho g H$$

$$\rightarrow \boxed{p_a = p_m + \frac{1}{2}\rho v_m^2}$$

$$\frac{1}{2}\rho v_m^2 = -(p_m - p_a) = \rho\frac{\Omega^2 R^2}{2} - \rho g H$$

$$v_m^2 = \Omega^2 R^2 - 2gH = \Omega^2 R^2\left(1 - \frac{2gH}{\Omega^2 R^2}\right)$$

$$\boxed{v_m = \Omega R\left(1 - \frac{2gH}{\Omega^2 R^2}\right)^{1/2}}$$

(3.) for effective pumping,

$$\frac{2gH}{\Omega^2 R^2} < 1 \qquad \boxed{\Omega^2 R^2 > 2gH}$$

(4.) minimum p occurs at max z and min x^2+y^2

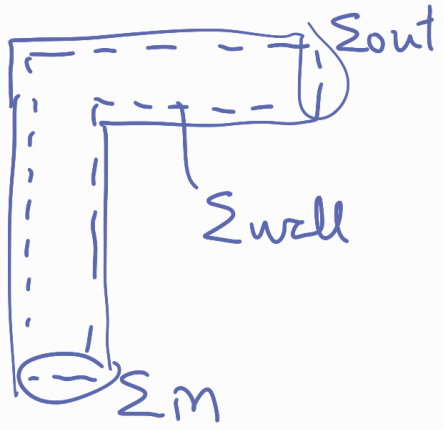
$\Rightarrow$ at $\bar{x}_0$. with $x = y = z = 0$

$$p_{min} = p_a - \rho\frac{\Omega^2 R^2}{2} > p_v$$

to avoid cavitation, $p_{min} > p_v$

$$\boxed{\Omega^2 R^2 < \frac{2(p_a - p_v)}{\rho}}$$

(5.) conservation of linear momentum



- rotating ref. frame.
- steady flow.
- ideal flow
- fixed C.V. in ($\bar{v}_c = 0$ on all Σ_c)

$$\int_{\Sigma_c} \rho \bar{v}\,(\bar{v}-\bar{v}_c)\cdot\bar{n}\,d\Sigma = -\int_{\Sigma_c} p\bar{n}\,d\Sigma + \int_{V_c} \rho \bar{f}_m\,dV$$

Σ_{in}: $\rho v_m \bar{e}_z (-v_m)\pi a^2$	Σ_{in}: $-p_m(-\bar{e}_z)\pi a^2$
Σ_{out}: $\rho v_m \bar{e}_x v_m \pi a^2$	Σ_{out}: $-p_a(\bar{e}_x)\pi a^2$
	Σ_w: $-\bar{F}$

($\bar{F}$: force exerted by the liquid on the pipe)

$$\bar{f}_m = \bar{g} - \bar{\Omega}\wedge(\bar{\Omega}\wedge\bar{x}) - 2\bar{\Omega}\wedge\bar{v}$$

Gravity force: $\bar{g}$

$$\int_{V_c} \rho\bar{g}\,dV = -\rho g\,\pi a^2 (H+R)\,\bar{e}_z$$

Centrifugal force: $-\bar{\Omega}\wedge(\bar{\Omega}\wedge\bar{x}) = \Omega^2 x\,\bar{e}_x$

- vertical arm $\rightsquigarrow 0$
- horizontal arm $\quad \bar{x} = x\,\bar{e}_x$
  $\bar{\Omega} = \Omega\,\bar{e}_z$

$$\bar{\Omega}\wedge\bar{x} = \begin{vmatrix} \bar{e}_x & \bar{e}_y & \bar{e}_z \\ 0 & 0 & \Omega \\ x & 0 & 0 \end{vmatrix} = \Omega x\,\bar{e}_y$$

$$\bar{\Omega}\wedge(\bar{\Omega}\wedge\bar{x}) = \begin{vmatrix} \bar{e}_x & \bar{e}_y & \bar{e}_z \\ 0 & 0 & \Omega \\ 0 & \Omega x & 0 \end{vmatrix} = -\Omega^2 x\,\bar{e}_x$$

$$\int_{x=0}^{x=R} \rho\,\Omega^2 x\,\bar{e}_x\,\pi a^2\,dx = \rho\,\Omega^2\,\frac{R^2}{2}\,\pi a^2\,\bar{e}_x$$

Coriolis force $\quad -2\,\bar{\Omega}\wedge\bar{v} = -2\Omega v_m\,\bar{e}_y$

- vertical arm $\rightsquigarrow 0$
- horizontal arm $\quad \bar{v} = v_m\,\bar{e}_x$
  $\bar{\Omega} = \Omega\,\bar{e}_z$

$$\bar{\Omega}\wedge\bar{v} = \begin{vmatrix} \bar{e}_x & \bar{e}_y & \bar{e}_z \\ 0 & 0 & \Omega \\ v_m & 0 & 0 \end{vmatrix} = \Omega v_m\,\bar{e}_y$$

$$\rho\int_{V_c} -2\Omega v_m\,\bar{e}_y\,dV = -2\rho\Omega v_m\,\pi a^2 R\,\bar{e}_y$$

Total mass forces:

$$\int_{V_c} \rho \bar{f}_m \, dV = -\rho g (H+R) \pi a^2 \bar{e}_z + \rho \frac{\Omega^2 R^2}{2} \pi a^2 \bar{e}_x - 2\rho\Omega v_m R \pi a^2 \bar{e}_y$$

Linear momentum conservation:

$$-\rho v_m^2 \pi a^2 \bar{e}_z + \rho v_m^2 \pi a^2 \bar{e}_x = \rho_m \pi a^2 \bar{e}_z - p_a \pi a^2 \bar{e}_x - \bar{F} - \rho g (H+R) \pi a^2 \bar{e}_z + \rho \frac{\Omega^2 R^2}{2} \pi a^2 \bar{e}_x - 2\rho\Omega v_m R \pi a^2 \bar{e}_y$$

$$\bar{F} = \left(-\rho v_m^2 - p_a + \rho \frac{\Omega^2 R^2}{2} \right) \pi a^2 \bar{e}_x - 2\rho\Omega R v_m \pi a^2 \bar{e}_y + \left(\rho v_m^2 + p_m - \rho g (H+R) \right) \pi a^2 \bar{e}_z$$

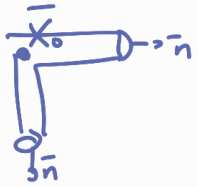
substitute in $\bar{F}$

$$v_m^2 = \Omega^2 R^2 - 2gH$$

$$\vdots$$

$$p_m = p_a + \rho g H - \rho \frac{\Omega^2 R^2}{2}$$

(5.) Angular momentum conservation

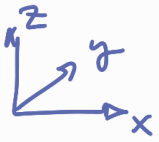


$$\int_{\Sigma_c} (\bar{x}-\bar{x}_0)\wedge \rho\bar{v}\,(\bar{v}-\bar{v}_c)\cdot\bar{n}\,d\sigma =$$

Σ_{in} 0
Σ_{out} 0
Σ_w 0

$$= -\int_{\Sigma_c} (\bar{x}-\bar{x}_0)\wedge p\bar{n}\,d\sigma + \int_{V_c} (x-\bar{x}_0)\wedge\rho\,\bar{f}_m\,dV$$

Σ_{in} 0
Σ_{out} 0
Σ_w $-\bar{n}_0$



Gravity force $\bar{g}$

- vertical arm : 0
- horizontal arm : $\bar{x}-\bar{x}_0 = x\,\bar{e}_x$

$\bar{g} = -g\,\bar{e}_z$

$$\int_{x=0}^{x=R} \rho g x\,\bar{e}_y\,\pi a^2\,dx \qquad \begin{vmatrix} \bar{e}_x & \bar{e}_y & \bar{e}_z \\ x & 0 & 0 \\ 0 & 0 & -g \end{vmatrix} = +gx\,\bar{e}_y$$

$$\rho g \frac{R^2}{2}\pi a^2\,\bar{e}_y$$

$$= \rho g \pi a^2 R\,\frac{R}{2}\,\bar{e}_y \quad \checkmark$$

Centrifugal force:

horizontal arm:

$$\int_{x=0}^{x=R} x\,\bar{e}_x \wedge \Omega^2 x\,\bar{e}_x\,\pi a^2\,dx = 0$$

Coriolis force:

horizontal arm:

$$\rho \int_{x=0}^{x=R} x\,\bar{e}_x \wedge (-2\Omega v_m)\bar{e}_y\,\pi a^2\,dx =$$

$$-\rho 2\Omega v_m \frac{R^2}{2}\,\pi a^2\,\bar{e}_z$$

$$\rightsquigarrow \boxed{\bar{M}_0 = \rho g \frac{R^2}{2}\pi a^2\,\bar{e}_y - 2\rho\Omega v_m \frac{R^2}{2}\pi a^2\,\bar{e}_z}$$

FLUID MECHANICS I **19-01-2022**
PROBLEM 3 (10 POINTS). **Time: 60 minutes**

NAME:..... **GRADE:.....**

The sketch below shows a micropumping device. The pipe of radius a has a vertical arm of length H and a horizontal arm of length $R \sim H \gg a$, and rotates with a **constant** angular velocity $\Omega\bar{e}_z$, and takes liquid from a low level reservoir. It is wished to characterize the flow rate Q in terms of the geometry (a, R, H), angular velocity (Ω) and the fluid properties (density ρ and viscosity μ).
In the anaysis, a reference system attached to the rotating pipe, shown in the sketch, is suggested (notice this is a rotating system, i.e. non inertial), where the reduced pressure P can be written as $P = p + \rho g z - \rho\Omega^2(x^2+y^2)/2$, in terms of the vertical distance z and the radial distance to the rotation axis $r = (x^2+y^2)^{1/2}$, to include both the gravity force and the centrifugal force. The Coriolis force can be neglected in the flow along the pipe.
Using the theory of flows dominted by viscosity,

1. Formulate the problem, with boundary conditions, that determines the longitudinal velocity v along the vertical part of the pipe, between points 1 and 0. Find v and the flow rate Q in terms of $\partial P/\partial z$, and determine $P(z)$ with Q as a parameter, using the boundary condition $P(z=-H) = p_a$. Determine $P_0 = P(z=0)$.

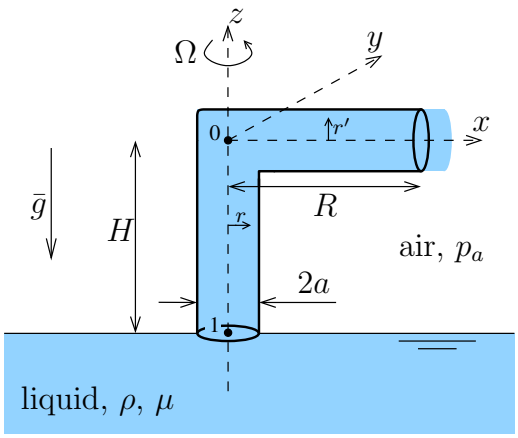
2. To determine the longitudinal velocity and pressure distribution along the horizontal arm of the pipe, it is suggested to use an auxiliar reference in cylindrical coordinate along the axis of the pipe, using x as the cylindrical axis and r' as the radial distance to this axis, as shown in the sketch. Use P_0 as boundary condition to determine $P(x)$.

3. Find the additional boundary condition for $P(x)$ that allows to close the problem and determine Q as a function of the parameters of the problem. Give the limits of Ω for effective pumping from the low-level reservoir to height H, and the corresponding limits of $\Lambda = 2gH/\Omega^2R^2$ and Q.

After determination of the flow rate, its limits will be refined taking into account cavitation. Cavitation is an undesirable phenomena in pumps that takes place when the static pressure decreases to a level equal to the vapor pressure of the liquid p_v, when bubbles are formed that reduce the performance of the pump. To this end, the point of minimum **static** pressure must be found,

4. Determine the **static** pressure along the vertical arm as a function of z, and determine the point where minimum pressure occurs and the value of pressure $p_{\min}$.

5. Find the critical angular velocity Ω_c above which cavitation takes place and find the maximum effective flow rate that can be obtained with this pump.



① $0 = -\frac{\partial P}{\partial z} + \frac{\mu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right)$

$r=a: v=0$

$r=0: \frac{\partial v}{\partial r}=0$

$$\Rightarrow \boxed{v = -\frac{1}{4\mu}\frac{\partial P}{\partial z}(a^2-r^2)}$$

$$\boxed{Q = \int_0^a -\frac{2\pi}{4\mu}\frac{\partial P}{\partial z}(a^2-r^2)\cdot r\,dr = -\frac{\pi a^4}{8\mu}\frac{\partial P}{\partial z}}$$

$$\left.\begin{aligned}\frac{\partial P}{\partial z} &= -\frac{8\mu Q}{\pi a^4}\\ z=-H:\ P &= P_a\end{aligned}\right\} \rightarrow \boxed{P = P_a - \frac{8\mu Q}{\pi a^4}(z+H)} \Rightarrow P_0 = P_a - \frac{8\mu Q}{\pi a^4}H$$

② SIMILARLY: ALONG X: $v = -\frac{1}{4\mu}\frac{\partial P}{\partial x}(a^2-x^2) \rightarrow Q = -\frac{\pi a^4}{8\mu}\frac{\partial P}{\partial x}$

$$\left.\begin{aligned}\frac{\partial P}{\partial x} &= -\frac{8\mu}{\pi a^4}Q\\ P(x=0) &= P_0\end{aligned}\right\} \boxed{P = P_0 - \frac{8\mu Q}{\pi a^4}x = P_a - \frac{8\mu Q}{\pi a^4}(H+x)}$$

③ AT $\left.\begin{matrix}z=0\\ x=R\end{matrix}\right\} P=P_a \Rightarrow P = P_a - \frac{\rho\Omega^2R^2}{2} = P_0 - \frac{8\mu Q}{\pi a^4}(H+R)$

$$\boxed{Q = \frac{\rho\Omega^2R^2}{2}\frac{\pi a^4}{8\mu(H+R)}} \qquad Q>0 \Rightarrow \Omega > 0$$

④ STATIC PRESSURE

$$P = P_1 + \rho g z - \frac{\rho\Omega^2(x^2+y^2)}{2} = P_a - \frac{8\mu Q}{\pi a^4}(z+H)$$

NEGLECT

$$P = P_a - \frac{\rho\Omega^2R^2}{\pi a^4}\frac{H}{H+R} - \underbrace{\rho g z - \frac{\rho\Omega^2R^2}{2}\frac{z}{H+R}}_{\text{DECREASING FUNCTION OF } z}$$

$$P_{min} = P(z=0) = P_a - \frac{\rho\Omega^2R^2}{2}\frac{H}{H+R}$$

⑤ CAVITATION: $P_{min} = P_v \rightarrow \boxed{\Omega^2 = (P_a - P_v)\,2\frac{(H+R)}{H}}$