FLUID MECHANICS I **21-06-2022**
THEORY (4 POINTS). **Time: 20 minutes**

NAME:..... **GRADE:.....**

QUESTION I

Consider the following 2D incompressible velocity field, given in cartesian coordinates: $\bar{v} = (u, v)$, where $u = Ax$, with A being a known constant.

1. Determine the y-component of the velocity field v, if a stagnation point exists at the origin and the flow is irrotational.
2. Determine the viscous stress tensor $\bar{\bar{\tau}}'$ at a generic position (x, y).
3. Find the pressure field $p(x, y)$ with the condition that the pressure at the stagnation point is 0.
4. Obtain the force exerted by the fluid to the right of segment OP, of length L, on the fluid to the left of this line.

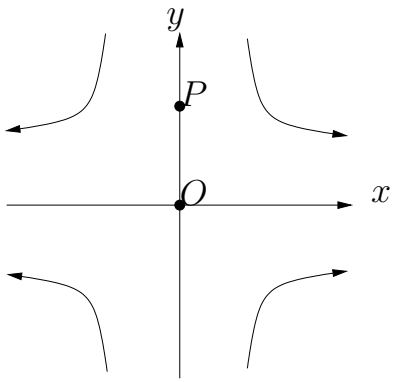
Note:

Continuity equation: $\partial\rho/\partial t + \nabla \cdot (\rho\bar{v}) = 0$

Momentum equation: $\rho\partial\bar{v}/\partial t + \rho\bar{v} \cdot \nabla\bar{v} = -\nabla p + \nabla \cdot \bar{\bar{\tau}}' + \rho\bar{f_m}$

Viscous stress tensor:

$$\bar{\bar{\tau}}' = \mu \begin{pmatrix} 2\partial u/\partial x & \partial u/\partial y + \partial v/\partial x \\ \partial u/\partial y + \partial v/\partial x & 2\partial v/\partial y \end{pmatrix} + (\mu_B - \frac{2}{3}\mu)\nabla \cdot \bar{v} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$



① $\nabla \cdot \vec{v} = 0 \Rightarrow A + \frac{\partial v}{\partial y} = 0 \Rightarrow v = -Ay + f(x)$

$\nabla \wedge \vec{v} = 0 \Rightarrow \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \frac{df}{dx} = 0 \Rightarrow f = C$

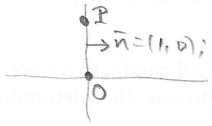
$\vec{v}(0,0) = (0,0) \Rightarrow C = 0 \rightarrow \boxed{v = -Ay}$

② $\bar{\bar{\tau}}' = \mu(\nabla\vec{v} + \nabla\vec{v}^T) + (\mu_B - \frac{2}{3}\mu)\nabla\cdot\vec{v}\,\bar{\bar{I}} = 2\mu A \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ ($\nabla\cdot\vec{v} = 0$)

③ $\rho\frac{\partial\vec{v}}{\partial t} + \rho\vec{v}\cdot\nabla\vec{v} = -\nabla p + \nabla\cdot\bar{\bar{\tau}}'$

$\left.\begin{aligned} \rho A^2 x &= -\frac{\partial p}{\partial x} + 0 \\ \rho A^2 y &= -\frac{\partial p}{\partial y} + 0 \end{aligned}\right\} \Rightarrow \boxed{p = \frac{\rho A^2}{2}(x^2+y^2)} + p_0$ ($p_0 = 0$)

④ $d\vec{F} = -p\vec{n}\,d\sigma + \bar{\bar{\tau}}'\cdot\vec{n}\,d\sigma$



$\left.\begin{aligned} d\vec{F} &= -\frac{\rho A^2}{2}(x^2+y^2)\,dy\,\vec{e}_x + 2\mu A\,dy\,\vec{e}_x \\ x &= 0 \text{ along OP} \end{aligned}\right\}$

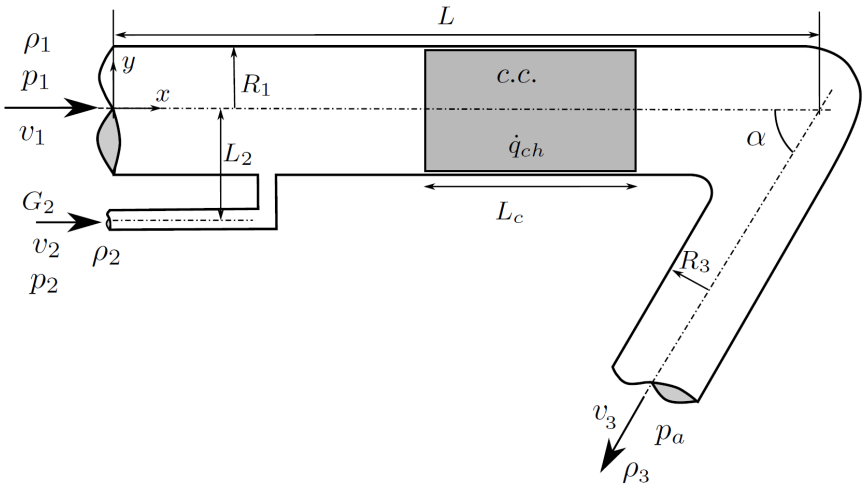
$\boxed{\vec{F} = \int_0^L d\vec{F} = \left(-\frac{\rho A^2}{6}L^3 + 2\mu AL\right)\vec{e}_x}$

ESCUELA POLITÉCNICA SUPERIOR

FLUID MECHANICS I **21-06-2022**
PROBLEM 1 (12 POINTS). **Time: 60 minutes**

NAME:..... **GRADE:.....**

The burner sketched in the figure is made of three different cylindrical channels: the main intake pipe with radius R_1, the fuel injector that is parallel to the main pipe but separated by a distance L_2, and the exhaust pipe with radius R_3 that is placed at a distance L from the intake and is orientated with an angle α, as shown in the sketch below. The properties of the inflow gas, namely density ρ_1, velocity v_1, and pressure p_1 are known, and so are the gas properties in the injector: mass flow rate G_2, density ρ_2, velocity v_2, and pressure p_2. Right after the mixing is done, the burning is carried out within the combustion chamber (c.c.) domain of length L_c and radius R_1. The chemical heat release per unit time and volume is $\dot{q}_{ch}$. Finally, the burnt gas is driven out through the exhaust pipe at atmospheric pressure p_a.



Assuming that **all walls are adiabatic**, that both fresh and burnt gases behave as a perfect gas with known specific heat ratio $\gamma = c_p/c_v$, that **gravity effects are negligible**, and that both inflow and outflow fluid properties are uniform, determine:

1. The expressions that allow obtaining the values of v_3 and ρ_3 at the exit and the injector radius R_2 as functions of known parameters only.
2. The **net force** $\vec{F}$ exerted by the internal gas and the atmosphere, on the burner.
3. The **net moment** $\vec{M}_O$ exerted by the gas on the burner with respect to the origin of the reference frame.
4. Is it possible to cancel the net moment by just changing the injector position L_2?

Make sure the line of reasoning is clear to the reader; in particular, adequate sketches are essential.

1. Mass conservation $\rho_1 v_1 A_1 + G_2 = \rho_3 v_3 A_3$ ($A_1 = \pi R_1^2$, $A_3 = \pi R_3^2$)

$G_2 = \rho_2 v_2 A_2$

$A_2 = \frac{\rho_2 v_2}{G_2} = \pi R_2^2$

Energy conservation (total energy)

$$\frac{d}{dt}\int_{V_c} \rho\left(e + \frac{v^2}{2}\right) dV + \int_{\Sigma_c} \rho\left(e + \frac{v^2}{2}\right) \bar{v}\cdot\bar{n}\, d\Sigma = -\int_{\Sigma_c} p\bar{n}\cdot\bar{v}\, d\Sigma + \int_{\Sigma_c} \bar{\bar{\tau}}\cdot\bar{n}\cdot\bar{v}\, d\Sigma + \int_{V_c} \rho\bar{g}\cdot\bar{v}\, dV - \int_{\Sigma_c} \bar{q}\cdot\bar{n}\, d\Sigma + \int_{V_c} Q_{rc}\, dV$$

Σ_c: Σ_1, Σ_2, Σ_3, $\Sigma_w \to 0$

- uniform inlet/outlet
- fixed walls (no moving parts)
- uniform inlet/outlet
- adiabatic walls

$\dot{q}_c V_{cc} = \dot{q}_c \pi R_1^2 L_c$

$$-\rho_1\left(e_1 + \frac{v_1^2}{2}\right) v_1 A_1 - \rho_2\left(e_2 + \frac{v_2^2}{2}\right) v_2 A_2 + \rho_3\left(e_3 + \frac{v_3^2}{2}\right) v_3 A_3 =$$

$\left(e_2 + \frac{v_2^2}{2}\right)$ multiplies G_2

$e_1 = f(\rho_1, \rho_1)$ ✓

$e_2 = f(\rho_2, \rho_2)$ ✓

$e = f(\rho_3, \rho_a)$?

$$= +\rho_1 v_1 A_1 + \underbrace{\rho_2 v_2 A_2}_{\frac{\rho_2 G_2}{\rho_2}} - \rho_a v_3 A_3 + \dot{q}_c \pi R_1^2 L_c$$

$$\rightsquigarrow -\rho v_1 A_1 \underbrace{\left(e_1 + \frac{p_1}{\rho_1} + \frac{v_1^2}{2}\right)}_{h_1 = \frac{\gamma}{\gamma-1}\frac{p_1}{\rho_1}} - G_2 \underbrace{\left(e_2 + \frac{p_2}{\rho_2} + \frac{v_2^2}{2}\right)}_{h_2 = \frac{\gamma}{\gamma-1}\frac{p_2}{\rho_2}} + \rho v_3 A_3 \underbrace{\left(e_3 + \frac{p_a}{\rho} + \frac{v_3^2}{2}\right)}_{h = \frac{\gamma}{\gamma-1}\frac{p_a}{\rho}} = \dot{q}_c \pi R_1^2 L_c$$

$e = c_v T + e_0$

$h = c_p T$

$h = e + \frac{p}{\rho}$

$p = \rho R_g T$

$\gamma = \frac{c_p}{c_v}$

$\Rightarrow$

$e = \frac{1}{\gamma-1}\frac{p}{\rho}$

$h = \frac{\gamma}{\gamma-1}\frac{p}{\rho}$

Linear momentum .-

$$\frac{d}{dt}\int_{V_c} \rho \bar{v}\, dV + \int_{\Sigma_c} \rho \bar{v}\,(\bar{v}-\bar{v}_c)\cdot \bar{n}\, d\sigma = -\int_{\Sigma_c} (p-p_a)\bar{n}\, d\sigma + \int_{\Sigma_c} \bar{\bar{\tau}}'\cdot \bar{n}\, d\sigma + \int_{V_c} \rho \bar{g}\, dV$$

Σ_c: Σ_1, Σ_2, Σ_3, $\Sigma_{wall} \to 0$

Σ_c: Σ_1, Σ_2, Σ_3, Σ_{wall} | $\Sigma_1 \sim 0$, $\Sigma_2 \sim 0$, $\Sigma_3 \sim 0$, Σ_{wall} $\to -\bar{F}$

$$\rho_1 v_1 \bar{e}_x (-v_1) A_1 + \rho_2 v_2 \bar{e}_x (-v_2) A_2 - \left(\rho_3 v_3 \cos\alpha\, \bar{e}_x + \rho_3 v_3 \sin\alpha\, \bar{e}_y\right) v_3 A_3 =$$

$$= +(p_1 - p_a)\,\bar{e}_x A_1 + (p_2 - p_a)\,\bar{e}_x A_2 - \bar{F}$$

$$\bar{F} = \left[+\rho v_1^2 A_1 + \rho_2 v_2^2 A_2 + \rho_3 v_3^2 \cos\alpha\, A_3 + (p_1 - p_a) A_1 + (p_2 - p_a) A_2\right] \bar{e}_x + \rho_3 v_3^2 \sin\alpha\, A_3\, \bar{e}_y$$

Angular momentum .-

$$\int_{\Sigma_c} (\bar{x}-\bar{x}_0)\wedge \rho \bar{v}\,(\bar{v}\cdot\bar{n})\, d\sigma = -\int_{\Sigma_c} (\bar{x}-\bar{x}_0)\wedge (p-p_a)\bar{n}\, d\sigma + \int_{\Sigma_c} (\bar{x}-\bar{x}_0)\wedge \bar{\bar{\tau}}'\cdot\bar{n}\, d\sigma$$

Σ_c: $\Sigma_1 \to 0$, Σ_2, $\Sigma_3 \to 0$

Σ_c: $\Sigma_1 \sim 0$, $\Sigma_2 \sim 0$, $\Sigma_3 \sim 0$, Σ_{wall} | Σ_{wall} $\to -\bar{M}_0$

$$\rho_2 v_2 L_2 (+\bar{e}_z)(-v_2) A_2 + \rho_3 v_3 L \sin\alpha\, (-\bar{e}_z) v_3 A_3 = -(p_2 - p_a) L_2 (-\bar{e}_z) A_2 - \bar{M}_0$$

$$\bar{M}_0 = \left[+\rho_2 v_2^2 A_2 L_2 + \rho_3 v_3^2 A_3 L \sin\alpha + (p_2 - p_a) A_2 L_2\right] \bar{e}_z$$

For $\bar{M}_0 = 0$ the injector should be placed opposite to the elbow

with $$L_2 = \frac{\rho_3 v_3^2 A_3 L \sin\alpha}{\rho_2 v_2^2 A_2 + (p_2 - p_a) A_2}$$



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FLUID MECHANICS I **21-06-2022**
PROBLEM 2 (10 POINTS). **Time: 60 minutes**

NAME:..... **GRADE:.....**

The sketch shows a circular cavity full of liquid of density ρ and viscosity μ surrounded by an outer case at temperature T_0. The fluid is initially at rest, temperature T_0 and pressure p_0. At $t = 0^+$, the outer case is impulsively set to rotate with a constant angular velocity Ω.

1. Use the Π theorem to reduce the parametric dependence of the torque M (per unit length) necessary to sustain the constant angular velocity of the outer case from $t = 0$ to $t \to \infty$.

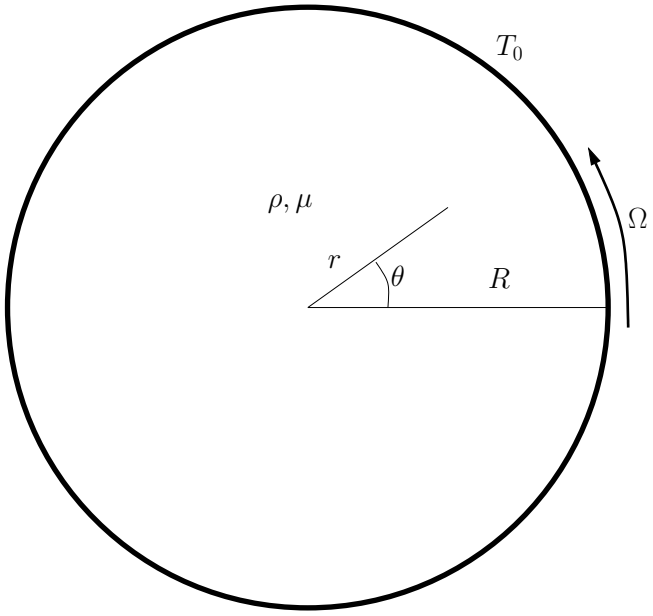
2. Formulate the **full** mathematical problem that determines the evolution of the fluid field (velocity, pressure and temperature) in the cavity $0 \leq r \leq R$ for $0 < t < \infty$, including boundary and initial conditions. Consider the pressure in the fluid adjacent to the wall is known to be p_1. Simplify the problem as much as possible to obtain the differential equations, with initial and boundary conditions, that determine the azimuthal velocity v_θ, T and p. Viscous dissipation might not be negligible.

3. Find v_θ, T and p in the case $t \to \infty$.

4. From the last result, obtain the adequate scales v_{θ_c}, r_c, respectively for v_θ, r. Find the adequate time scale t_c using order-of-magnitude analysis in the time-dependent differential equation for v_θ. Make the problem non-dimensional introducing the scales in the dimensional problem.

5. Define the expression that needs to be evaluated to find the torque M at any instant of time t, using the detailed solution of the non-dimensional problem, scaling M adequately.

6. Compare the last result with that of part #1 and comment the result



Note: It is strongly recommended to use cylindrical coordinates

① $M = M(t, S, \mu, \Omega, R)$

	M	L	T
M	1	1	−2
t	0	0	1
S	1	−3	0
μ	1	−1	−1
Ω	0	0	−1
R	0	1	0

TORQUE PER UNIT LENGTH

$\det \neq 0 \Rightarrow \mu, \Omega, R$ ARE D.I

PI THEOREM: $5-3 = 2$ PARAMETERS

$\Pi_0 = \frac{M}{\mu \Omega R^2}$

$\Pi_1 = \Omega t$

$\Pi_2 = \frac{S \Omega R^2}{\mu}$

$\rightarrow \boxed{\Pi_0 = f(\Pi_1, \Pi_2)}$

② CONTINUITY: $\frac{1}{r}\frac{\partial}{\partial r}(r V_r) + \frac{1}{r}\frac{\partial V_\theta}{\partial \theta} = 0$ (1) $\rightarrow$ $rV_r = f(t) = 0 \Rightarrow \boxed{V_r = 0}$

($V_\theta = 0$ SYMMETRY; $r = R: V_r = 0$)

MOMENTUM-r: $S\frac{\partial V_r}{\partial t} + SV_r\frac{\partial V_r}{\partial r} + S\frac{V_\theta}{r}\frac{\partial V_r}{\partial \theta} - S\frac{V_\theta^2}{r} = -\frac{\partial P}{\partial r} + \mu\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rV_r)\right)$ (2)

($V_r = 0$; $\frac{\partial V_r}{\partial r} = 0$; $\frac{\partial V_r}{\partial \theta} = 0$; $\frac{\partial}{\partial r}(rV_r) = 0$)

MOMENTUM-θ: $S\frac{\partial V_\theta}{\partial t} + SV_r\frac{V_\theta}{r} + S\frac{V_\theta}{r}\frac{\partial V_\theta}{\partial \theta} + SV_r\frac{V_\theta}{r} = -\frac{1}{r}\frac{\partial P}{\partial \theta} + \mu\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rV_\theta)\right)$ (3)

($V_r = 0$; $\frac{\partial V_\theta}{\partial \theta} = 0$; $\frac{\partial P}{\partial \theta} = 0$)

ENERGY: $Sc\frac{\partial T}{\partial t} + ScV_r\frac{\partial T}{\partial r} + Sc\frac{V_\theta}{r}\frac{\partial T}{\partial \theta} = \Phi_v + \frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)$ (4)

($V_r = 0$; $\frac{\partial T}{\partial \theta} = 0$)

$\Phi_v = \mu\left[r\frac{\partial}{\partial r}\left(\frac{V_\theta}{r}\right)\right]^2$

SIMPLIFYING:

$V_r = 0$ (1)'

$S\frac{V_\theta^2}{r} = \frac{\partial P}{\partial r}$ (2)'

$\frac{\partial V_\theta}{\partial t} = \nu\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rV_\theta)\right)$ (3')

$Sc\frac{\partial T}{\partial t} = \frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \mu\left[r\frac{\partial}{\partial r}\left(\frac{V_\theta}{r}\right)\right]^2$ (4')

I.C: $t = 0$: $V_\theta = 0$; $T = T_0$

B.C: $r = R$: $V_\theta = \Omega R$; $T = T_0$

$r = 0$: $V_\theta = 0$; $\frac{\partial T}{\partial r} = 0$

③ $t \to \infty \Rightarrow \frac{\partial}{\partial t} = 0$: (3') $\Rightarrow \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rV_\theta)\right) = 0 \rightarrow \frac{1}{r}\frac{\partial}{\partial r}(rV_\theta) = A \rightarrow rV_\theta = \frac{Ar^2}{2} + B$

$V_\theta = \frac{A}{2}r + \frac{B}{r}$ ($B = 0$: NO SINGULARITY) $\Rightarrow \boxed{V_\theta = \Omega r}$

$r = R: V_\theta = \Omega R$

(2') $\Rightarrow \frac{S\Omega^2 r^2}{r} = \frac{\partial P}{\partial r} \Rightarrow P = \frac{S\Omega^2 r^2}{2} + C' \xrightarrow{P(R) = P_1}$ $\boxed{P = P_1 + \frac{1}{2}S\Omega^2R^2\left[\left(\frac{r}{R}\right)^2 - 1\right]}$

(4') $\Rightarrow \Phi_v = 0 \Rightarrow \frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) = 0 \rightarrow r\frac{\partial T}{\partial r} = D \rightarrow T = D\ln r + E \rightarrow \boxed{T = T_0}$

($D = 0$: NO SINGULARITY)

④ $v_{\theta_c} = \Omega R$, $r_c = R$

$$\frac{\partial v_\theta}{\partial t} = \nu \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r v_\theta)\right)$$

$$\frac{v_{\theta_c}}{t_c} \sim \nu \frac{v_{\theta_c}}{r_c^2} \rightarrow \underline{t_c \sim \frac{R^2}{\nu}}$$

USING THESE SCALES:

$V = \frac{v_\theta}{\Omega R}$, $\tau = \frac{t}{R^2/\nu}$, $\eta = \frac{r}{R}$ $\rightarrow$

$$\boxed{\frac{\partial V}{\partial \tau} = \frac{\partial}{\partial \eta}\left[\frac{1}{\eta}\frac{\partial}{\partial \eta}(\eta V)\right]}$$

$\tau = 0$: $V = 0$

$\tau > 0$: $\eta = 1$: $V = 1$

$\eta = 0$: $V = 0$

PROBLEM WITHOUT PARAMETERS.

THE SAME CAN BE DONE WITH T:

$$T = \frac{T_\theta}{T_0} = \Theta \rightarrow \frac{\nu \rho c}{R^2} T_0 \frac{\partial \Theta}{\partial \tau} = \frac{k T_0}{R^2}\frac{1}{\eta}\frac{\partial}{\partial \eta}\left[\eta \frac{\partial \Theta}{\partial \eta}\right] + \frac{\mu \Omega^2 R^2}{R^2}\left[\eta \frac{\partial}{\partial \eta}\left(\frac{V}{\eta}\right)\right]^2$$

$$\boxed{\frac{\partial \Theta}{\partial t} = \frac{Pr}{\eta}\frac{\partial}{\partial \eta}\left[\eta \frac{\partial \Theta}{\partial \eta}\right] + \frac{\Omega^2 R^2}{c T_0}\left[\eta \frac{\partial}{\partial \eta}\left(\frac{V}{\eta}\right)\right]^2}$$

$\tau = 0$: $\Theta = 0$

$\tau > 0$: $\eta = 1$: $\Theta = 0$

$\eta = 0$: $\frac{\partial \Theta}{\partial \eta} = 0$

$Pr = \frac{\mu c}{k}$

$\frac{\Omega^2 R^2}{c T_0}$: NON-DIMENSIONAL PARAMETER

⑤ $M = \tau_{r\theta}\big|_{r=R} \cdot R \cdot 2\pi R$; $\tau_{r\theta} = \mu\left[r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \mu \underbrace{\frac{\partial v_r}{\partial \theta}}_{0}\right] = \frac{\mu \Omega R}{R}\eta \frac{\partial}{\partial \eta}\left(\frac{V}{\eta}\right)$

$$M = 2\mu \Omega R^2 \pi \underbrace{\left[\eta \frac{\partial}{\partial \eta}\left(\frac{V}{\eta}\right)\right]_{\eta=1}}_{f(\tau)} \Rightarrow \frac{M}{2\pi \mu \Omega R^2} = f(\tau)$$

NOTICE THAT $\tau \rightarrow \infty \Rightarrow V \rightarrow \eta \rightarrow f(\tau \rightarrow \infty) \rightarrow 0 \Rightarrow \frac{M}{2\pi\mu\Omega R^2} \xrightarrow[\tau \rightarrow \infty]{} 0$

⑥ COMPARING WITH #1 : $\pi_0 = f\left(\underbrace{\frac{t\nu}{R^2}}_{\pi_1'}\right)$ $\underline{\pi_1' = \frac{\pi_1}{\pi_2}}$ CONSISTENT.

ESCUELA POLITÉCNICA SUPERIOR

FLUID MECHANICS I **21-06-2022**
PROBLEM 3 (10 POINTS). **Time: 60 minutes**

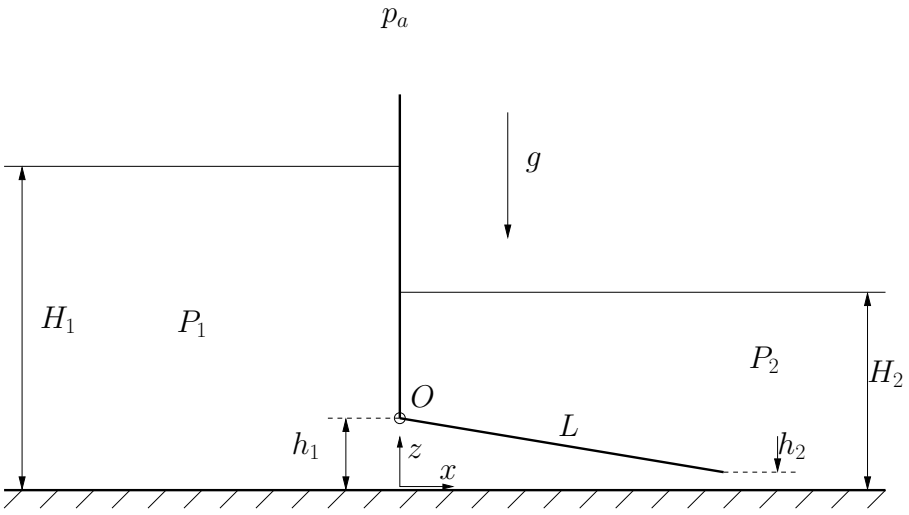
NAME:..... **GRADE:.....**

The sketch below shows two large reservoirs of liquid of density ρ and viscosity μ and heights H_1 and H_2 respectively, separated by a hinged planar gate of mass m and length L. The hinge is located at point O, at a height $h_1 \ll L$ above the floor, and allows free rotation of the gate.
We will consider first the **fluidstatics** case, when the free end of the gate is touching the floor ($h_2 = 0$),

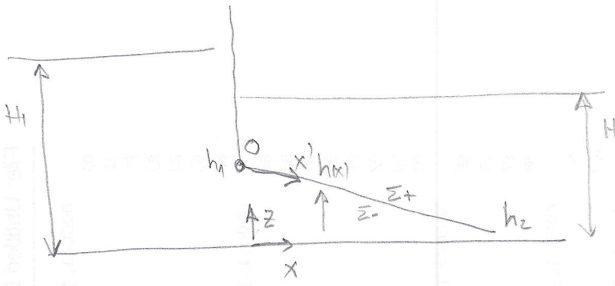
1. Find the difference $H_1 - H_2$ necessary for the gate to open and connect both reservoirs.

Assuming the above condition is satisfied, the liquid is transfered from one reservoir to the other in a flow **dominated by the viscosity**. In order to determine the final height h_2 of the free end of the gate, it is suggested to proceed as indicated next: Take h_2 as a known magnitude and solve the whole problem. Then, an additional condition must be found to determine h_2 (or β),

2. Find the velocity profile $u(z)$ as a function of the reduced pressure gradient $\partial P/\partial x$ at a generic position x along the channel between the floor and the gate
3. Obtain the flow rate Q (yet unknown) as a function of the pressure gradient
4. Find the reduced pressure $P(x)$ along the channel and determine Q
5. Obtain the equilibrium condition that determines h_2



Note: The height of the channel below the gate can be written as $h(x) = h_1(1 - \beta x/L)$, for $0 \leq x \leq L$ with $\beta = (h_1 - h_2)/h_1$.



$$h(x) = h_1 - \frac{(h_1-h_2)}{L}x = h_1\left[1-\beta\frac{x}{L}\right] \qquad \beta = \frac{h_1-h_2}{h_1} < 1$$

① FLUID STATICS: $\Sigma \vec{M}_0 = 0$

$$\vec{M}_0 = \int_0^L (P_- - P_+)x'dx' - mg\frac{L}{2} = 0$$

$P_- = P_1 - \rho_g h_1(x)$; $P_1 = P_a + \rho_g H_1$

$P_+ = P_2 - \rho_g h_2(x)$; $P_2 = P_a + \rho_g H_2$

$$= \int_0^L \rho_g (H_1-H_2)x'dx' - mg\frac{L}{2} = \rho_g(H_1-H_2)\frac{L^2}{2} - mg\frac{L}{2} = 0 \Rightarrow \boxed{H_1-H_2 > \frac{m}{\rho L}}$$ GATE OPENS.

② $0 = -\frac{\partial P}{\partial x} + \mu\frac{\partial^2 u}{\partial z^2}$

$u(z=0)=0$

$u(z=h(x))=0$

$$\Rightarrow \boxed{u = -\frac{1}{2\mu}\frac{\partial P}{\partial x}z(h-z)}$$

③ $$\boxed{Q = \int_0^{h(x)} u\,dz = -\frac{1}{2\mu}\frac{\partial P}{\partial x}\left[h\frac{z^2}{2} - \frac{z^3}{3}\right]_0^h = -\frac{h^3(x)}{12\mu}\frac{\partial P}{\partial x}}$$

④ Continuity $\Rightarrow$ Q is not a function of x

$$\left.\begin{aligned}\frac{\partial P}{\partial x} &= -\frac{12\mu}{h^3(x)}Q \\ P(x=0) &= P_1\end{aligned}\right\} \Rightarrow P(x) \rightarrow \text{USING } P(x=L) = P_2 \Rightarrow Q \text{ IS DETERMINED.}$$

$$P - P_1 = -12\mu Q\int_0^x \frac{dx}{h^3(x)} = -\frac{12\mu QL}{h_1^3}\int_0^\xi \frac{d\xi}{(1-\beta\xi)^3} = \frac{-12\mu QL}{h_1^3}\cdot\frac{1}{2\beta}\left[\frac{1}{(1-\beta\xi)^2}\right]_0^\xi$$

$\xi = \frac{x}{L}$

$$= -\frac{6\mu QL}{h_1^3\beta}\left[\frac{1}{(1-\beta\frac{x}{L})^2} - 1\right]$$

NOW: $P(x=L)=P_2 \Rightarrow$ $P_2 - P_1 = -\frac{6\mu QL}{h_1^3\beta}\left[\frac{1}{(1-\beta)^2}-1\right] \Rightarrow \boxed{Q = \frac{\rho_g(H_1-H_2)h_1^3\beta}{6\mu L\left[\frac{1}{(1-\beta)^2}-1\right]}}$

SO THAT $$\boxed{P(x) - P_1 = -\frac{\rho_g(H_1-H_2)}{\left[\left(\frac{1}{1-\beta}\right)^2-1\right]}\cdot\left[\frac{1}{(1-\beta\frac{x}{L})^2}-1\right]}$$

(5) h_2 (AND THEREFORE β) IS DETERMINED BY EQUILIBRIUM OF GATE.

$$\overset{\curvearrowleft}{\Sigma M_0} = 0 = \int_0^L (P - P_T)\, x\, dx - mg\frac{L}{2} = 0$$

$$\left.\begin{aligned} P &= P(x) - \rho g h(x) \\ P_T &= P_2 - \rho g h(x) \end{aligned}\right\} \qquad \int_0^L \left(P(x) - P_2\right) x\, dx - mg\frac{L}{2} = 0$$

$$\left(P_1 - P_2\right)\frac{L^2}{2} - mg\frac{L}{2} + \underbrace{\int_0^L \frac{-\rho g(H_1 - H_2)}{\left[\frac{1}{(1-\beta\xi)^2} - 1\right]} \cdot \left[\frac{1}{\left(1 - \beta\frac{x}{L}\right)^2} - 1\right] x\, dx}_{I} = 0$$

$$I = -\frac{\rho g(H_1 - H_2) L^2}{\left(\frac{1}{1-\beta}\right)^2 - 1} \int_0^1 \xi \left[\frac{1}{(1-\beta\xi)^2} - 1\right] d\xi = \frac{-\rho g(H_1 - H_2) L^2}{\left(\frac{1}{1-\beta}\right)^2 - 1} \left[-\frac{1}{2} + \frac{1}{\beta^2}\ln(1-\beta) + \frac{1}{\beta^2}\left(\frac{1}{1-\beta} - 1\right)\right]$$

$$= \frac{-\rho g(H_1 - H_2) L^2}{\left(\frac{1}{1-\beta}\right)^2 - 1} \cdot \left[-\frac{1}{2} + \frac{1}{\beta^2}\ln(1-\beta) + \frac{1}{\beta(1-\beta)}\right]$$

SO THE EQUATION BECOMES:

$$\boxed{\frac{\rho g(H_1 - H_2) L^2}{2}\left[1 - \frac{1}{\left(\frac{1}{1-\beta}\right)^2 - 1}\left(-1 + \frac{2}{\beta^2}\ln(1-\beta) + \frac{2}{\beta(1-\beta)}\right)\right] - mg\frac{L}{2} = 0}$$

$$\Downarrow$$

$$\beta = \frac{h_1 - h_2}{h_1} \Rightarrow \underline{\underline{h_2}}$$