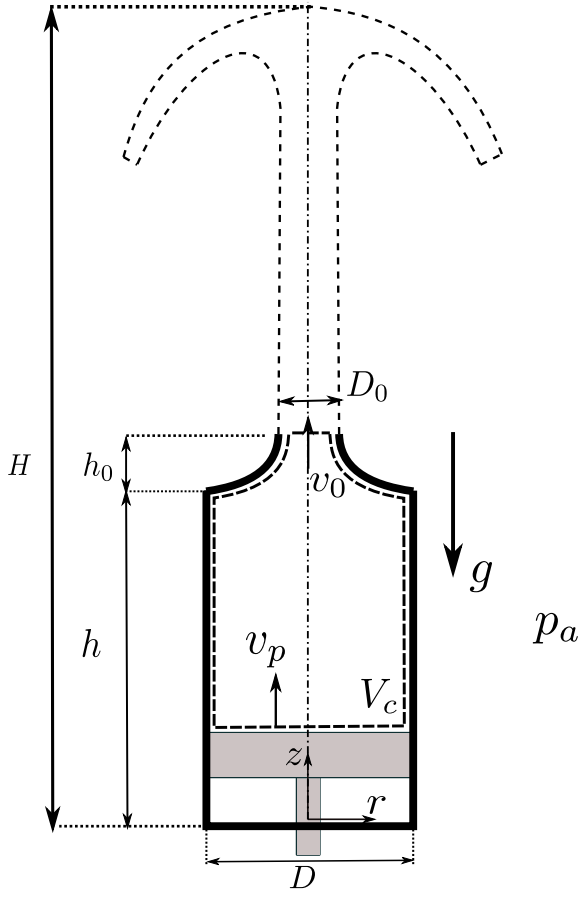
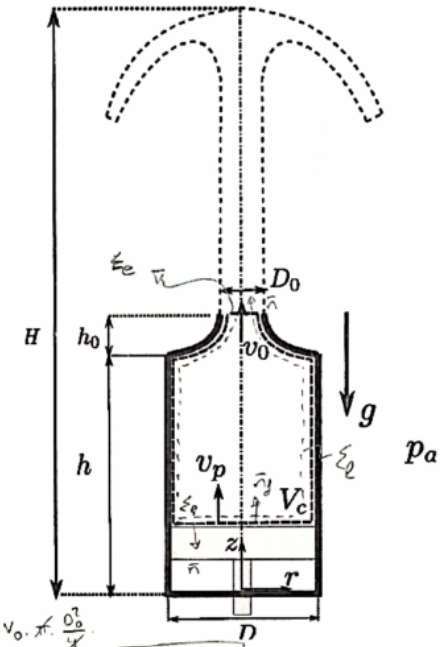
FLUID MECHANICS I **18-01-2021**
PROBLEM 1 (10 POINTS). **Time: 80 minutes**

NAME:..... **GRADE:.....**

The sketch below represents a piston pump with initial volume $V_0 = \pi(D/2)^2 h$ (the volume of the nozzle is very small and can be neglectd $h_0 D_0^2 \ll hD^2$). The piston moves upwards with unknown velocity v_p squirting liquid out of the cylinder with constant uniform velocity v_0 through a nozzle with minimum diameter D_0. With this information, assuming that the viscous forces can be neglected ($Re = \rho v_0 D_0/\mu \gg 1$) and using the control volume V_c indicated in the figure

1. Obtain the velocity v_p of the piston.
2. Calculate the force exerted by the liquid on the pump (piston + walls).
3. Calculate the power needed to eject the liquid and the force exerted by the fluid on the piston.
4. Obtain the maximum height H reached by the fluid





1) $\frac{d}{dt}\left(\cancel{\pi}\cdot\frac{D^2}{\cancel{4}}\cdot(h-z)\cdot\rho\right) = -v_0\cdot\cancel{\pi}\cdot\frac{D_0^2}{\cancel{4}}$

$D^2\cdot\frac{dz}{dt} = v_0\cdot D_0^2 \Rightarrow \boxed{v_p = v_0\cdot\left(\frac{D_0}{D}\right)^2}$

2) $\bar{F} = -\int_{\xi_p+\xi_\ell}(p-p_a)\,\bar{n}_y\,d\tau \qquad \xi_c = \xi_p + \xi_\ell + \xi_e$

Momentum equation: $\frac{d}{dt}\int_{V_c}\rho\cdot\bar{v}\,dV + \int_{\xi_e}\rho\cdot\bar{v}(\bar{v}-\bar{v}_e)\,\bar{n}\,d\tau = -\int_{\xi_p+\xi_\ell}(p-p_a)\,\bar{n}\,d\tau + \int_{V_c}\rho\,\bar{f}_m\,dV$

$\left(\frac{d}{dt}\left(\rho\,v_p\,\pi\,\frac{D^2}{4}(h-z)\right) + \rho\cdot v_0^2\cdot\pi\,\frac{D_0^2}{4} = -\int_{\xi_p+\xi_\ell}(p-p_a)\,\bar{n}\,d\tau - \rho\,g\cdot\pi\,\frac{D^2}{4}\cdot(h-z)\right)\bar{e}_z$

$\left(-\pi\,\frac{D^2}{4}\,v_p^2 + \pi\,\frac{D_0^2}{4}\cdot v_0^2 + \pi\,\frac{D^2}{4}\,g\,(h-z) = -\int_{\xi_p+\xi_\ell}(p-p_a)\,\bar{n}\,d\tau = -\frac{F}{\rho}\right)\bar{e}_z$

$$\boxed{\pi\,\frac{D_0^2}{4}\,v_0^2\left(1-\left(\frac{D_0}{D}\right)^2\right) + \pi\,\frac{D^2}{4}\,g\,(h-z) = -\frac{F}{\rho}}$$

3) $\dot{W} = -\int_{\xi_p}(p-p_a)\cdot\bar{v}\,\bar{n}\,d\sigma$

Energy equation: $\frac{d}{dt}\int_{V_c}\rho\cdot\left(e+\frac{v^2}{2}\right)dV + \int_{\xi}\rho\cdot\left(e+\frac{v^2}{2}\right)(\bar{v}-\bar{v}_e)\,\bar{n}\,d\tau = -\int_{\xi}(p-p_a)\,\bar{v}\,\bar{n}\,d\tau + \int_{V_c}\rho\cdot\bar{f}_m\cdot\bar{v}\,dV$

$\frac{d}{dt}\left(\rho\cdot\left(e+\frac{v^2}{2}\right)\cdot\pi\,\frac{D^2}{4}\cdot(h-z)\right) + \rho\left(e+\frac{v_0^2}{2}\right)\cdot v_0\cdot\pi\,\frac{D_0^2}{4} = -\int_{\xi_p}(p-p_a)\,\bar{v}\,\bar{n}\,d\sigma - \rho\cdot g\cdot v_z\cdot\pi\,\frac{D^2}{4}\,(h-z)$

$-\rho\cdot\left(\cancel{e}+\frac{v_p^2}{2}\right)\pi\,\frac{D^2}{4}\,v_p + \rho\left(\cancel{e}+\frac{v_0^2}{2}\right)\cdot\pi\,\frac{D_0^2}{4}\cdot v_0 + \rho\,g\,\frac{\pi\,D^2}{4}\cdot(h-z)\,v_p = \dot{W}$

$$\boxed{\frac{\pi}{8}\,D_0^2\,v_0^3\cdot\left(1-\left(\frac{D_0}{D}\right)^4\right) + g\cdot\frac{\pi\cdot D_0^2}{4}\,v_0\,(h-z) = \frac{\dot{W}}{\rho}}$$

$$\boxed{\bar{F}_p = \frac{\dot{W}\,\bar{e}_z}{v_p} = -\rho\pi\frac{D^2}{8}\,v_0^2\left(1-\left(\frac{D_0}{D}\right)^4\right) - \rho g\,\frac{\pi D^2}{4}\,(h-z)\cdot\bar{e}_z}$$

4) Bernoulli's equation at exit

$\cancel{p_a} + \frac{1}{2}\rho\,v_0^2 + \rho g\,(h+h_0) = \cancel{p_a} + \rho\,g\,H$

$$\boxed{H = \frac{v_0^2}{2g} + h\left(1+\frac{h_0}{h}\right) \simeq \frac{v_0^2}{2g} + h}$$

FLUID MECHANICS I **18-01-2021**
PROBLEM 2 (10 POINTS). **Time: 80 minutes**

NAME:..... **GRADE:.....**

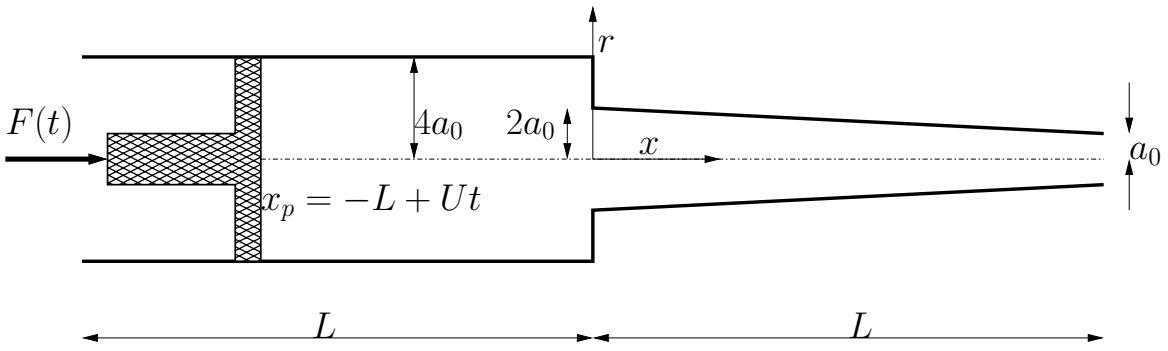
The sketch below shows a slender syringe made of a cylinder of radius $4a_0$ and length $L >> a_0$ and a thin conical needle of length L and radius $a(x) = 2a_0\left(1 - \frac{x}{2L}\right)$. A piston of mass M pushes the liquid of density ρ and viscosity μ, that discharges in the atmosphere at a pressure p_a. If the piston is initially located at $x_p(0) = -L$ and moves to the right with a constant velocity U,

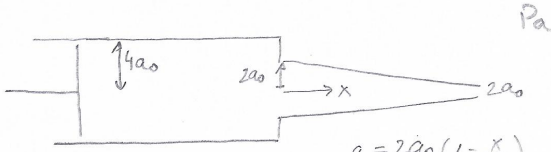
1. Give the criteria for the flow to be dominated by viscosity. Assume in what follows that this criterion is satisfied. **(2 points)**

2. Determine the force $F(t)$, the power $W(t)$ and the total energy E needed to move the piston, as specified above, to the end of the cylinder. Show that the force exerted by the liquid on the piston can be written, approximately as $F_p \simeq \alpha U$, and find the constant α. **(6 points)**

3. Assume now that instead of moving the piston with constant velocity the piston is moved from rest by a constant force F_0. Determine the power $W(t)$, the piston velocity $U(t)$ and the total energy E in this case. Use the approximation abtained above. **(2 points)**

Note: Neglect the effect of gravity





$$a = 2a_0\left(1 - \frac{x}{2L}\right)$$

(1) FLOW DOMINATED BY VISCOSITY:

- UNSTEADY/VISCOUS $\ll 1$.

$$\frac{\frac{\rho U_c}{t_c}}{\frac{\mu U_c}{a_0^2}} = \frac{\rho a_0^2}{\mu t_c} = \frac{\rho U a_0}{\mu}\cdot\frac{a_0}{L} \ll 1. \qquad t_c = \frac{L}{U}$$

- CONVECTION/VISCOUS $\ll 1$

$$\frac{\frac{\rho U_c^2}{L}}{\frac{\mu U_c}{a_0^2}} = \frac{\rho U_c a_0}{\mu}\frac{a_0}{L} = \frac{\rho U a_0}{\mu}\cdot\frac{a_0}{L} \ll 1.$$

} SAME!

(2) Need $P(x_p)$; $x_p = -(L - Ut)$; TWO INTERVALS : $x_p < x < 0$; $0 < x < L$

2.1 :

$$0 = -\frac{\partial P}{\partial x} + \frac{\mu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) \rightarrow \frac{\partial u}{\partial r} = \frac{1}{\mu}\frac{\partial P}{\partial x}\cdot\frac{r}{2} + \frac{\cancel{A}}{r} \rightarrow u = \frac{r^2}{4\mu}\frac{\partial P}{\partial x} + B$$

$\frac{\partial u}{\partial r} = 0$ at $r = 0$

$u = 0$ at $r = a(x)$

$$0 = \frac{a^2}{4\mu}\frac{\partial P}{\partial x} + B$$

$$\boxed{u = -\frac{1}{4\mu}\frac{\partial P}{\partial x}\left(a^2(x) - r^2\right)}$$

2.2 :

$$Q = \int_0^a 2\pi r u(r)\,dr = -\frac{2\pi}{4\mu}\frac{\partial P}{\partial x}\left[a^2\frac{r^2}{2} - \frac{r^4}{4}\right]_0^a \Rightarrow \boxed{Q = -\frac{\pi a^4(x)}{8\mu}\frac{\partial P}{\partial x}}$$

2.3 : Continuity $\Rightarrow$ $\boxed{Q = 16\pi a_0^2 U = \text{const.}}$

2.4 : $\frac{\partial P}{\partial x} = -\frac{8\mu}{\pi a^4(x)}Q$; $P(x=L) = P_a$

TWO CASES: $x > 0$ AND $x < 0$.

- $x > 0$:

$$-P(x) = \frac{8\mu Q}{\pi 16 a_0^4}\int\frac{dx}{\left(1 - \frac{x}{2L}\right)^4} = \frac{\mu Q}{2\pi a_0^4}\cdot\frac{1}{\left(1 - \frac{x}{2L}\right)^3}\cdot\frac{2L}{3} + A$$

$$-P(x) = \frac{\mu Q L}{3\pi a_0^4}\frac{1}{\left(1 - \frac{x}{2L}\right)^3} + A$$

$$-P_a = \frac{\mu Q L}{3\pi a_0}\frac{1}{(1/2)^3} + A$$

$$\rightarrow \boxed{P(x) - P_a = \frac{\mu Q L}{3\pi a_0^4}\left[8 - \frac{1}{\left(1 - \frac{x}{2L}\right)^3}\right]}$$

$$P(0) = P_a = \frac{7\mu Q L}{3\pi a_0^4}$$

- $x_p < x < 0$: $\frac{\partial P}{\partial x} = -\frac{8\mu Q}{\pi(4a_0)^4} = -\frac{\mu Q}{32\pi a_0^4} \rightarrow P(x) = -\frac{\mu Q}{32\pi a_0^4}x + B$

$P(0) =$ B

$$P(x) - P_a = \frac{7\mu Q L}{3\pi a_0^4} - \frac{\mu Q x}{32\pi a_0^4}$$

2.5 :

$$F_p = \left(P(x_p) - P_a\right)\pi(4a_0)^2(-\bar{e}_x) = -16\pi a_0^2\left[\frac{7\mu Q L}{3\pi a_0^4} - \frac{\mu Q x_p}{32\pi a_0^4}\right]\bar{e}_x$$

$$F = -F_p = \frac{16\pi a_0^2 \mu Q L}{\pi a_0^4}\left[\frac{7}{3} - \frac{x_p}{32L}\right]$$

Using $\alpha = 16\pi a_0^2 U \rightarrow F(t) = \frac{16\pi a_0^2 \mu L}{\pi a_0^2} \cdot 16\pi a_0^2 U \left[\frac{7}{3} - \frac{Xp}{32L}\right]$

$$\boxed{F(t) = 256\pi\mu UL \left[\frac{7}{3} + \frac{L-Ut}{32L}\right] \simeq 256\pi\mu UL \cdot \frac{7}{3}}$$

$\frac{L-Ut}{32L} \ll \frac{7}{3}$

$$\boxed{W(t) = F(t) \cdot U = 256\pi\mu U^2 L \left[\frac{7}{3} + \frac{L-Ut}{32L}\right]}$$

$$E = \int_0^{L/U} W(t)\,dt = 256\pi\mu U^2 L \left[\frac{7}{3}\frac{L}{U} + \frac{1}{32}\frac{L}{U} - \frac{\cancel{0}}{\cancel{32}}\left(\frac{L}{U}\right)^{\cancel{2}} \frac{1}{2}\right]$$

$$\underline{E} = 256\pi\mu UL^2 \left[\frac{7}{3} + \frac{1}{32} - \frac{1}{2}\cdot\frac{1}{32}\right] = \underline{256\pi\mu UL^2 \frac{448}{192}}$$

(3) Now $U(t)$, to be determined.

$$\begin{cases} M\frac{dU}{dt} = F_0 - F_p \simeq F_0 - \frac{1792}{3}\pi\mu U L\left[1 - \underbrace{\frac{3X_p}{224L}}_{\ll 1}\right] \simeq F_0 - \alpha \cdot U \\ U(0) = 0 \end{cases} \qquad \alpha = \frac{1792}{3}\pi\mu L$$

$$\frac{dU}{F_0 - \alpha U} = \frac{1}{M}dt \rightarrow \left.-\frac{\ln(F_0 - \alpha U)}{\alpha}\right]_0^U = \frac{t}{M}$$

$$\ln F_0 - \ln(F_0 - \alpha U) = \frac{\alpha}{M}t \rightarrow \ln\left|\frac{F_0 - \alpha U}{F_0}\right| = -\frac{\alpha t}{M} \rightarrow 1 - \frac{\alpha}{F_0}U = e^{-\frac{\alpha t}{M}}$$

$$\boxed{U(t) = \frac{F_0}{\alpha}\left[1 - e^{-\frac{\alpha t}{M}}\right]}$$

$$W(t) = F_0 \cdot U(t) = \frac{F_0^2}{\alpha}\left[1 - e^{-\frac{\alpha t}{M}}\right]$$

$$E = F_0 \cdot L$$

FLUID MECHANICS I **18-01-2021**
PROBLEM 3 (5 POINTS). **Time: 40 minutes**

NAME:..... **GRADE:.....**

A pendulum is made of a cable of length L and negligible mass and diameter and a hanging spherical mass m of radius $R \ll L$. It evolves in time from rest and an initial angle θ_0. The movement takes place in an incompressible fluid of density ρ and viscosity μ. If the instantaneous position θ and the critical viscosity μ_c, above which the movement of the pendulum is not oscillatory, are to be studied in the case of large viscosity,

1. Determine the parametric dependence of the position θ and μ_c. Check that, for θ, L, g and μ are dimensionally independent and, for μ_c, L, g and ρ are also dimensionally independent, and use the Π theorem to reduce both parametric dependences.

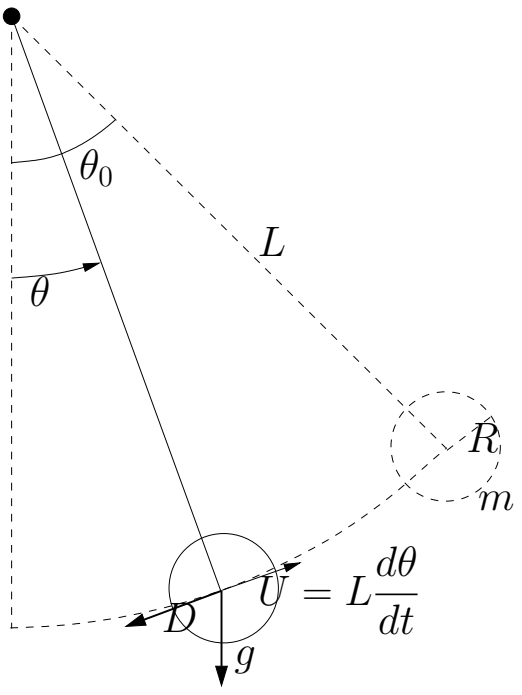
Next, a mathematical model of the sphere's movement will be developed as a more physical way to find the relevant parameters of the problem. It is suggested to follow the following procedure:

2. Show that the drag experienced by a sphere moving with a velocity U in a stagnant fluid can be written as $D = \frac{1}{2}\rho U^2 \pi R^2 C_D(Re)$, with $Re = \frac{\rho U R}{\mu}$, where U is the velocity of the sphere relative to the fluid and C_D is the drag coefficient. .

3. Find an explicit expression for the drag in the limit $Re \ll 1$, when $C_D(Re) \simeq \frac{12}{Re}$. Formulate the mathematical problem that describes the evolution of θ in time for smal Reynolds numbers, using the expression for drag obtained above. Non-dimensionalize it and find the relevant physical parameters that determine θ. It is sugested to non-dimensionalize time as $\tau = t\sqrt{g/L}$.

4. Compare the expressions found with those obtained in #1. Simplify in the case $\theta_0 \ll 1$.

5. Determine the relevant parameters in the case $Re \gg 1$, taking into account that, in this case, $C_D(Re) \simeq C_{D_0} = \text{const}$.



① $\theta = \theta(t, \theta_0, m, L, R, \rho, \mu, g) \rightarrow \theta = \theta\left(t\sqrt{\frac{g}{L}}, \theta_0, \frac{m}{\rho L^3}, \frac{R}{L}, \frac{\rho g^{1/2} L^{3/2}}{\mu}\right)$ (1)

$\mu_c = \mu_c(\theta_0, m, L, R, \rho, g) \rightarrow \frac{\mu_c}{\rho g^{1/2} L^{3/2}} = f\left(\theta_0, \frac{m}{\rho L^3}, \frac{R}{L}\right)$ (2)

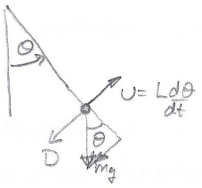
② SPHERE:

$\rightarrow \rho$
$\rightarrow \mu$ (1/R)
$\rightarrow U$

$D = D(\rho, \mu, U, R) \rightarrow \frac{D}{\rho U^2 R^2} = f_2(R_0) \Leftrightarrow D = \frac{1}{2}\rho U^2 \pi R^2 C_D(R_0)$

IF $R_e \ll 1$: $C_D = \frac{24}{R_0} \rightarrow D = \frac{1}{2}\rho U^2 \pi R^2 \frac{24\mu}{\rho U 2R} = 6\pi\mu U R$

③ GENERIC POSITION:



TANGENTIAL MOVEMENT:

$$mL\frac{d^2\theta}{dt^2} = -mg\sin\theta - 6\pi\mu R L\frac{d\theta}{dt}$$

USING $\tau = t\sqrt{\frac{g}{L}}$ →

$$\begin{cases} mL\frac{g}{L}\frac{d^2\theta}{d\tau^2} + mg\sin\theta + 6\pi\mu R L\sqrt{\frac{g}{L}}\frac{d\theta}{d\tau} \\ \theta(0) = \theta_0 \\ \frac{d\theta}{d\tau}(0) = 0 \end{cases}$$

DIVIDING BY mg: $\frac{d^2\theta}{d\tau^2} + \sin\theta + \frac{6\pi\mu R}{m}\sqrt{\frac{L}{g}}\frac{d\theta}{d\tau} = 0$, $\theta(0) = \theta_0$; $\frac{d\theta}{d\tau}(0) = 0$ →

$$\boxed{\frac{d^2\theta}{d\tau^2} + \sin\theta + \Lambda\frac{d\theta}{d\tau} = 0, \quad \theta(0) = \theta_0;\ \frac{d\theta}{d\tau}(0) = 0}$$

④ THEREFORE: $\theta = \theta(\tau; \theta_0, \Lambda)$, $\Lambda = 6\pi\frac{\mu}{\rho\sqrt{g}L^{3/2}} \cdot \frac{\rho L^3}{m} \cdot \frac{R}{L}$, ENCOMPASSING 3 PARAMETERS IN (1).

DAMPING IS CHARACTERIZED BY Λ. A CRITICAL VALUE EXISTS $\Lambda_c(\theta_0)$ ABOVE WHICH NO OSCILLATORY MOVEMENT EXISTS

$$6\pi\frac{\mu}{\rho g^{1/2}L^{3/2}} \cdot \frac{\rho L^3}{m} \cdot \frac{R}{L} = \Lambda_c(\theta_0) \rightarrow \boxed{\mu_c = \frac{\rho g^{1/2} L^{3/2}}{6\pi}\frac{m}{\rho L^3} \cdot \frac{L}{R}\Lambda_c(\theta_0)}$$

IF $|\theta| < \theta_0 \ll 1$, $\sin\theta \simeq \theta \Rightarrow$ $\frac{d^2\theta}{d\tau^2} + \Lambda\frac{d\theta}{d\tau} + \theta = 0$, $\theta(0) = \theta_0$; $\frac{d\theta}{d\tau}(0) = 0$ → $\varphi = \frac{\theta}{\theta_0}$: $\frac{d^2\varphi}{d\tau^2} + \Lambda\frac{d\varphi}{d\tau} + \varphi = 0$, $\varphi(0) = 1$; $\frac{d\varphi}{d\tau}(0) = 0$

THE CHARACTERISTIC EQN.: $\lambda^2 + \Lambda\lambda + 1 = 0 \rightarrow \lambda = \frac{-\Lambda \pm \sqrt{\Lambda^2 - 4}}{2}$

NON-OSCILLATORY MOVEMENT IF $\Lambda > \Lambda_c = 2$

THE SOLUTION: $\varphi = \varphi(\tau; \Lambda) \rightarrow \theta = \theta_0 \cdot \varphi(\tau; \Lambda)$

NOTICE THAT MORE PHYSICAL PARAMETERS CAN BE USED:

- $\frac{m}{\rho R^3}$ INSTEAD OF $\frac{m}{\rho L^3}$
- $\frac{\rho\sqrt{gL}\cdot R}{\mu}$ INSTEAD OF $\frac{\rho\sqrt{gL}\cdot L}{\mu}$, BECAUSE THE CHARACTERISTIC VELOCITY CAN BE EXPECTED TO BE, AT MOST, $V_c \sim \sqrt{gL(1-\cos\theta_0)} \sim \sqrt{gL}$, AND THE REYNOLDS NUMBER IS TYPICALLY BASED ON RADIUS (OR DIAMETER).

SO THAT (1) CAN BE WRITTEN AS

$$f\left(t\sqrt{\frac{g}{L}}, \theta_0, \frac{m}{\rho L^3}, \frac{R}{L}, \frac{\rho g^{1/2}L^{3/2}}{\mu}\right) = f\left(t\sqrt{\frac{g}{L}}, \theta_0, \frac{m}{\rho R^3}\left(\frac{R}{L}\right)^3, \frac{R}{L}, \frac{\rho\sqrt{gL}}{\mu}R\cdot\frac{L}{R}\right)$$

$$= \tilde{f}\left(t\sqrt{\frac{g}{L}}, \theta_0, \frac{m}{\rho R^3}, \frac{R}{L}, \frac{\rho\sqrt{gL}R}{\mu}\right)$$

AND $\Lambda = 6\pi\frac{\mu}{\rho\sqrt{gL}R}\frac{\rho R^3}{m}\cdot\frac{L}{R}$

⑤ IF $R_e \gg 1$:

$$mL\frac{d^2\theta}{dt^2} = -mg\sin\theta - \frac{1}{2}\rho L^2\frac{d\theta}{dt}\left|\frac{d\theta}{dt}\right|\pi R^2 C_{D_0}$$

$$\theta(0) = \theta_0$$

$$\frac{d\theta}{dt}(0) = 0$$

$$\Rightarrow \quad \frac{d^2\theta}{d\tau^2} + \sin\theta + \frac{\pi C_{D_0}}{2}\frac{\rho R^3}{m}\cdot\frac{L}{R}\frac{d\theta}{d\tau}\left|\frac{d\theta}{d\tau}\right| = 0$$

$$\theta(0) = \theta_0; \quad \frac{d\theta}{d\tau}(0) = 0$$

THE PARAMETER IS $\xi = \frac{\pi C_{D_0}}{2}\frac{\rho R^3}{m}\frac{L}{R}$

IF $\xi \ll 1$, $\theta_0 \ll 1$. THE SIMPLE PENDULUM IS RECOVERED:

$$\frac{d^2\varphi}{d\tau^2} + \varphi = 0 \qquad \varphi(0) = 1; \frac{d\varphi}{d\tau}(0) = 0$$

$$\varphi = \cos\tau$$