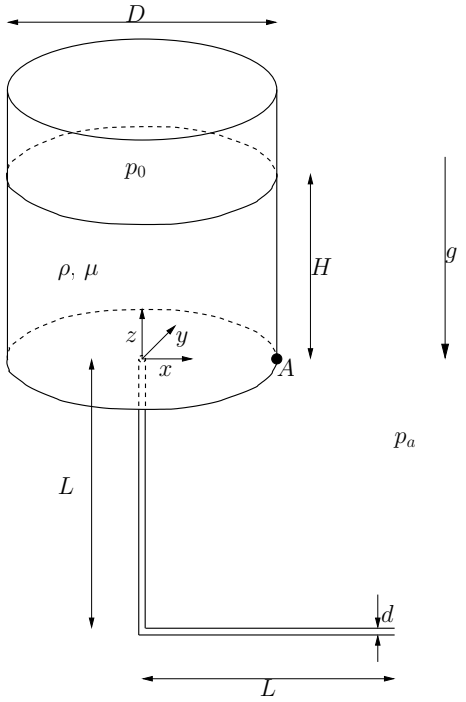
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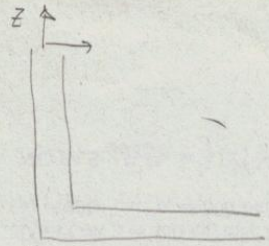
INTRODUCTION TO FLUID MECHANICS. **17-01-2017**
PROBLEM 1 (10 POINTS). **Time: 50 minutes**

NAME:..... **GROUP:.....**

The sketch shows a cylindrical tank and al L-shaped pipe. The tank contains, initially, a liquid of density ρ and viscosity μ, up to a level H, and a gas pressurized to a pressure p_0, that will remain constant in the discharge process to be analyzed. Starting at time $t = 0$, the liquid discharges in the open atmosphere through the exit of the pipe and the level $h(t)$ of liquid in the tank will decrease until the complete discharge of the liquid in the tank. In the analysis to be carried out, the viscous effects can be neglected.

1. Find the velocity at the exit v_s and the volumentric flow rate Q. Consider the flow of the liquid in the pipe to be steady.
2. In order to justify the previous analysis, write the momentum equation (in fifferential form) projected in the direction of the streamline and, using order of magnitude analysis, show the criteria to be satisfied for the unsteady term to be negligible when compared with the convective term.
3. Formulate the problem, with initial conditions as needed, that determines the evolution of h as a function of time.
4. Determine the discharge time $t = t_{\mathrm{d}}$
5. Formulate the integral that would yield the total force $\bar{F}$ exerted by the fluid on the tank and the pipe. Everything must be clearly and completely specified either mathematically or graphically (sketch).
6. Find $\bar{F}$ exerted by the fluid on the tank and the pipe. Indicate **clearly** the control volume used in the analysis. Consider the unsteady term to be negligible and justify, using order of magnitude estimates, comparing the unsteady and convective term in the integral equation.
7. Obtain the total moment, about point A, exerted by the fluid on the tank and the pipe. Neglect the unsteady term.



① 

Viscosity negligible
Steady
ρ = const.
→ Bernoulli

$$P_0 + \rho g h = P_a - \rho g L + \frac{1}{2}\rho V_s^2 \rightarrow \boxed{V_s = \sqrt{\frac{2(P_0 - P_a)}{\rho} + 2g(L + h(t))}}$$

$$\boxed{Q = \pi\frac{d^2}{4} V_s = \pi\frac{d^2}{4}\sqrt{\frac{2(P_0 - P_a)}{\rho} + 2g(L + h(t))}}$$

② $$\rho\frac{\partial V}{\partial t} + \rho\frac{\partial V^2/2}{\partial z} = -\frac{\partial}{\partial z}(P + \rho g z)$$

$$t_d \sim \frac{H}{V_s\left(\frac{d}{D}\right)^2}$$

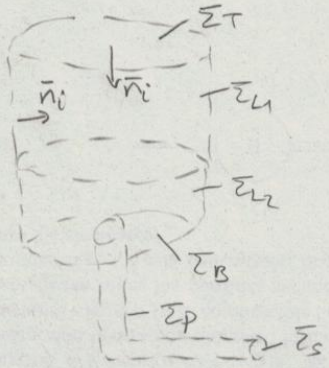
ρV_s^2 / t_d ; ρV_s^2 / L

$$\rightarrow \frac{O\left(\rho\frac{\partial V}{\partial t}\right)}{O\left(\rho\frac{\partial V^2/2}{\partial z}\right)} \sim \frac{\rho V_s / t_d}{\rho V_s^2 / L} \sim \frac{L}{t_d V_s} \sim \frac{L\, d^2}{H D^2} = \frac{V_{pipe}}{V_{TANK}} \ll 1.$$

③ $$\frac{\pi D^2}{4}\frac{dh}{dt} = -Q = -\frac{\pi d^2}{4}\sqrt{\frac{2(P_0 - P_a)}{\rho} + 2g(L+h)}$$

$$\begin{cases} \frac{dh}{dt} = -\left(\frac{d}{D}\right)^2\sqrt{\frac{2(P_0 - P_a)}{\rho} + 2g(L+h)} \\ h(0) = H \end{cases}$$

④ $$\int_H^0 \frac{dh}{\sqrt{\frac{2(P_0-P_a)}{\rho} + 2g(L+h)}} = \int_0^{t_d} \left(\frac{d}{D}\right)^2 dt \rightarrow \boxed{t_d = \left(\frac{D}{d}\right)^2\sqrt{2}\sqrt{\frac{L}{g}}\left[\sqrt{\frac{P_0 - P_a}{\rho g L} + 1 + \frac{H}{L}} - \sqrt{\frac{P_0 - P_a}{\rho g L} + 1}\right]}$$

⑤ 

$$\bar{F} = \int_{\Sigma_T+\Sigma_{L1}+\Sigma_{L2}+\Sigma_B+\Sigma_P} -(P - P_a)\bar{n}_i\, d\sigma + \int_{\Sigma_T+\Sigma_{L1}+\Sigma_{L2}+\Sigma_B+\Sigma_P} \bar{\bar{\tau}}\cdot\bar{n}_i\, d\sigma$$

$\bar{n}_i$: INWARD POINTING NORMAL VECTOR.

⑥ $$\frac{d}{dt}\int_{V_C}\rho\bar{v}\,dV + \int_{\Sigma_L}\rho\bar{v}(\bar{v} - \bar{v}_c)\cdot\bar{n}\, d\sigma = \int_{\Sigma_C} -P\bar{n}\, d\sigma + \int_{\Sigma_C}\bar{\bar{\tau}}\cdot\bar{n}\, d\sigma + \int_{V_C}\rho\bar{g}\, dV$$

$\bar{n}$: OUTWARD POINTING NORMAL VECTOR

TANK: $\frac{\rho V_c D^2 H}{t_d}$; PIPE: $\frac{\rho V_s d^2 L}{t_d}$; $\rho V_s^2 d^2$

$$V_c \sim V_s\cdot\left(\frac{d}{D}\right)^2$$

$$t_d \sim \frac{H}{V_c} \sim \frac{H D^2}{V_s d^2}$$

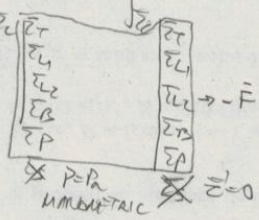
$$\frac{\rho V_c D^2 H / t_d}{\rho V_s^2 d^2} \sim \frac{\rho V_s \left(\frac{d}{D}\right)^2 D^2 H\, V_s d^2}{\rho V_s^2 d^2 \cdot H D^2} \sim \left(\frac{d}{D}\right)^2 \ll 1.$$

$$\frac{\rho V_s d^2 L / t_d}{\rho V_s^2 d^2} \sim \frac{\rho V_s^2 d^2 L\, V_s D}{\rho V_s^2 d^2 \cdot H D^2} \sim \frac{d^2 L}{D^2 H} \sim \frac{V_{PIPE}}{V_{TANK}} \ll 1.$$

IF BOTH CRITERIA ARE MET:

$$\int_{S_2} \rho \bar{v}(\bar{v}-\bar{v}_c)\cdot\bar{n}\,dS = \int_{S_2} -p\bar{n}\,dS + \int_{S_2} \bar{\bar{\tau}}\cdot\bar{n}\,dS = \int_{V_c} \rho \bar{g}\,dV$$

$\bar{v}=\bar{v}_c=0$



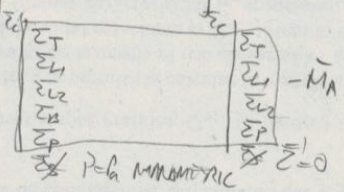
$$\rho V_s^2 \pi \frac{d^2}{4}\bar{e}_x = -\bar{F} + \rho g\left[\pi\frac{D^2}{4}h + \pi\frac{d^2}{4}\cdot 2L\right](-\bar{e}_z)$$

$$\boxed{\bar{F} = -\rho V_s^2 \pi\frac{d^2}{4}\bar{e}_x - \rho g \pi\frac{d^2}{4}L\left[\left(\frac{D}{d}\right)^2\frac{h}{L} + 2\right]\bar{e}_z}$$

(7) $\frac{d}{dt}\int_{V_c}(\bar{x}-\bar{x}_A)\times\rho\bar{v}\,dV + \int_{S_2}(\bar{x}-\bar{x}_A)\times\rho\bar{v}(\bar{v}-\bar{v}_c)\cdot\bar{n}\,dS = \int_{S_2}(\bar{x}-\bar{x}_A)\times(-p\bar{n})\,dS + \int_{S_2}(\bar{x}-\bar{x}_A)\times(\bar{\bar{\tau}}\cdot\bar{n})\,dS + \int_{V_c}(\bar{x}-\bar{x}_A)\times\rho\bar{g}\,dV$

QUASI-STEADY

$\bar{v}-\bar{v}_c=0$



$$\rho V_s^2 \pi\frac{d^2}{4}L(-\bar{e}_y) = -\bar{M}_A + \underbrace{\rho g \pi\frac{D^2}{4}h\frac{D}{2}(-\bar{e}_y) + \rho g \pi\frac{d^2}{4}L\frac{D}{2}(-\bar{e}_y)}_{\ll} + \rho g\pi\frac{d^2}{4}L\left(\frac{L}{2}-\frac{D}{2}\right)\bar{e}_y$$

$$\boxed{\bar{M}_A = \rho V_s^2 \pi\frac{d^2}{4}L - \rho g\,\pi\frac{D^2}{4}h\frac{D}{2} + \rho g\,\pi\frac{d^2}{4}L\left(\frac{L-D}{2}\right)\bar{e}_y}$$

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PROBLEM 2 (10 POINTS). **Time: 50 minutes**

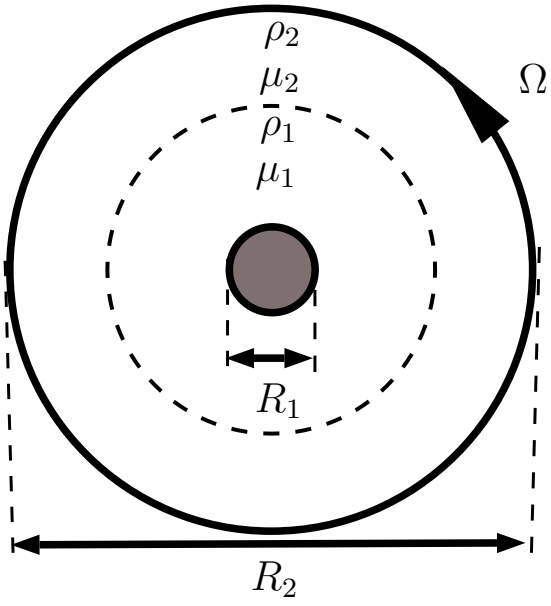
NAME:..... **GROUP:.....**

In an industrial application there are two concentric cylinders separated by a layer of two immiscible fluids. The interface separating fluids 1 and 2 is located at $r = (R_1 + R_2)/2$. If the outer cylinder spins with angular velocity Ω,

1. write the condition that ensures that the normal force at the interface $r = (R_1 + R_2)/2$ is equal in both fluids.

2. calculate the tangential velocity v_θ for the fluids 1 and 2. To do so, use the momentum equation for every fluid and specify, properly, the boundary conditions that determine the integration constants at the surface of the inner and outer cylinders and at the interface between the two fluids.

3. compute the pressure distribution in both fluids giving, in particular, the pressure at the interface of both fluids if the pressure $p = p_a$ at $r = R_2$.



a) $\left.p\cdot\bar{n}+\bar{\tau}\cdot\bar{n}\right|_1 = \left.p\cdot\bar{n}+\bar{\tau}\cdot\bar{n}\right|_2$

$\bar{\tau}=\begin{pmatrix}\tau_{rr} & \tau_{r\theta}\\ \tau_{r\theta} & \tau_{\theta\theta}\end{pmatrix} \rightarrow \bar{\tau}\cdot\bar{n}=\begin{pmatrix}\tau_{rr} & \tau_{r\theta}\\ \tau_{r\theta} & \tau_{\theta\theta}\end{pmatrix}\cdot\begin{pmatrix}1\\0\end{pmatrix} = \tau_{rr}\cdot\bar{e}_r + \tau_{r\theta}\cdot\bar{e}_\theta$

$\bar{n} = \bar{e_r}$

$$\boxed{(p_1+\tau_{rr_1})\cdot\bar{e}_r + \tau_{r\theta_1}\,\bar{e}_\theta = (p_2+\tau_{rr_2})\cdot\bar{e}_r + \tau_{r\theta_2}\cdot\bar{e}_\theta}$$

b) Momentum equation θ:

$$0=\mu\cdot\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial\theta}(r\cdot v_\theta)\right) \rightarrow \quad v_{\theta_2} = A_1\cdot\frac{r}{2}+\frac{A_2}{r}$$

$$v_{\theta_1} = B_1\frac{r}{2}+\frac{B_2}{r}$$

Boundary conditions:

$r=R_1 \rightarrow v_{\theta_1}=0 \Rightarrow$ $\boxed{0=B_1\cdot\frac{R_1}{2}+\frac{B_2}{R_1}}$ (1)

$r=R_2 \rightarrow v_{\theta_2}=\Omega \Rightarrow$ $\boxed{\Omega = A_1\cdot\frac{R_2}{2}+\frac{A_2}{R_2}}$ (2)

$r=\frac{R_1+R_2}{2} \rightarrow v_{\theta_1}=v_{\theta_2} \Rightarrow$ $\boxed{A_1\cdot\frac{R_m}{2}+\frac{A_2}{R_m}=B_1\cdot\frac{R_m}{2}+\frac{B_2}{R_m}}$ (3)

$\tau_{r\theta_1}=\tau_{r\theta_2} \rightarrow \mu_1\cdot r\cdot\frac{\partial}{\partial r}\left(\frac{v_{\theta_1}}{r}\right)=\mu_2\, r\frac{\partial}{\partial r}\left(\frac{v_{\theta_2}}{r}\right)$

$$\boxed{A_2=\frac{\mu_1}{\mu_2}\cdot B_2} \quad (4)$$

Solving the system of equations:

$$A_1=2\cdot\frac{\Omega}{R_2}\left\{1-\frac{1}{1-\left(\frac{R_2}{R_m}\right)^2+\frac{\mu_2}{\mu_1}\left[\left(\frac{R_2}{R_m}\right)^2-\left(\frac{R_2}{R_1}\right)^2\right]}\right\}$$

$$B_2=\frac{\Omega\cdot R_2}{\frac{\mu_1}{\mu_2}\cdot\left(1-\left(\frac{R_2}{R_m}\right)^2\right)+\left(\frac{R_2}{R_m}\right)^2-\left(\frac{R_2}{R_1}\right)^2}$$

$$B_1=-\frac{2\cdot B_2}{R_1^2}$$

$$A_2=\frac{\mu_1}{\mu_2}\cdot B_2$$

c) Radial momentum equation

$$\rho \cdot \frac{v_\theta^2}{r} = \frac{\partial p}{\partial r} \quad \Rightarrow \quad \frac{p_1}{\rho} = \frac{A_1 \cdot r^2}{8} - \frac{A_2^2}{2 \cdot r^2} + A_1 \cdot A_2 \cdot \ln r + \frac{p_{01}}{\rho}$$

$$\frac{p_2}{\rho} = \frac{B_1 \cdot r^2}{8} - \frac{B_2^2}{2 \cdot r^2} + B_1 B_2 \cdot \ln r + \frac{p_{02}}{\rho}$$

At $r = R_2$, $p_2 = p_a$ $\Rightarrow$ $p_{02} = p_a - \rho \left[\frac{B_1 \cdot R_2^2}{8} - \frac{B_2^2}{2 R_2^2} + B_1 \cdot B_2 \ln R_2 \right]$

At the interface $r = R_m$ $\quad p_1 = p_2 \rightarrow$ $p_{01} = p_{02} + \rho \cdot \left[\frac{B_1 - A_1}{8} R_m^2 - \frac{B_2^2 - A_2^2}{2 \cdot R_m^2} + \ln R_m \cdot \left(B_1 B_2 - A_1 A_2 \right) \right.$

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INTRODUCTION TO FLUID MECHANICS. **17-01-2017**
PROBLEM 3 (10 POINTS). **Time: 50 minutes**

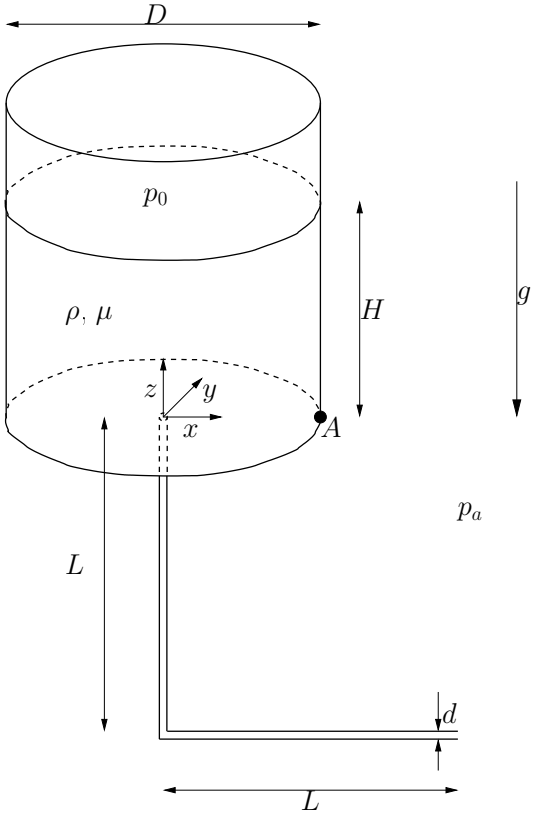
NAME:..... **GROUP:.....**

The sketch shows a cylindrical tank and al L-shaped pipe. The tank contains, initially, a liquid of density ρ and viscosity μ, up to a level H, and a gas pressurized to a pressure p_0, that will remain constant in the discharge process to be analyzed. Starting at time $t = 0$, the liquid discharges in the open atmosphere through the exit of the pipe and the level $h(t)$ of liquid in the tank will decrease until the complete discharge of the liquid in the tank, at $t = t_\mathrm{d}$. In the discharge process, the fluids exert a force F_x, on the tank and on the pipe, in the x-direction. This force will be a function of t through $h(t)$.

1. Find the (dimensional) parametric dependence of $t = t_\mathrm{d}$ and F_x.
2. Determine the maximum number of dimensionally independent variables and use the Pi theorem to reduce the number of parameters.
3. Simplify the results for the case of negligible viscous effects ($Re \to \infty$).

The system is to be installed as a water reservoir in Mars with $P_0 = 2 \times 10^5\,\mathrm{Pa}$, $D = 10\,\mathrm{mm}$ $H = 4\,\mathrm{m}$, $d = 2\,\mathrm{cm}$, $L = 50\,\mathrm{m}$. In order to ensure it works correctly, a scale model –hereafter referred to with subindex 2– is to be built and tested in an Earth-based facility where the maximum size is limited to $L_2 = 1\,\mathrm{m}$.

4. Design the experiment to be carried out in the facility. If, in the experiment, a discharge time $t_{\mathrm{d},2} = 10\,\mathrm{h}$ is obtained, What would be the discharge time in the real system? Check the validity of the analysis.



NOTE: $g_\mathrm{Mars} = 3.7\,\mathrm{m/s^2}$, $\rho_\mathrm{water} = 1000\,\mathrm{kg/m^3}$, $\mu_\mathrm{water} = 10^{-3}\mathrm{kg/(m \cdot s)}$, $P_{a,\mathrm{Mars}} = 600\,\mathrm{Pa}$

① $t_d = t_d(\rho, \mu, g, P_0 - P_a, D, H, d, L)$

$F_x = F_x(\rho, \mu, g, P_0 - P_a, D, H, d, L)$

②

	M	L	T
F_x	1	1	-2
t_d	0	0	1
ρ	1	-3	0
μ	1	-1	-1
g	0	1	-2
$P_0 - P_a$	1	-1	-2
D	0	1	0
H	0	1	0
d	0	1	0
L	0	1	0

rank = 3, ⇒ 3 D.I variables

ρ, g, L D.I, ADEQUATE FOR NEGLIGIBLE VISCOSITY

$$\begin{vmatrix} 1 & -3 & 0 \\ 0 & 1 & -2 \\ 0 & 1 & 0 \end{vmatrix} \neq 0$$

PI-THEOREM: $\pi_{t_d} = \frac{t_d}{\sqrt{L/g}}$; $\pi_{F_x} = \frac{F_x}{\rho g L^3}$

$\pi_\mu = \frac{\rho g^{1/2} L^{3/2}}{\mu}$; $\pi_{P_0 - P_a} = \frac{P_0 - P_a}{\rho g L}$, $\pi_D = \frac{D}{L}$, $\pi_H = \frac{H}{L}$, $\pi_d = \frac{d}{L}$

$$\frac{t_d}{\sqrt{L/g}} = f_1\left(\frac{\rho g^{1/2} L^{3/2}}{\mu}, \frac{P_0 - P_a}{\rho g L}, \frac{D}{L}, \frac{H}{L}, \frac{d}{L}\right) ; \frac{F_x}{\rho g L^3} = f_2\left(\frac{\rho g^{1/2} L^{3/2}}{\mu}, \frac{P_0 - P_a}{\rho g L}, \frac{D}{L}, \frac{H}{L}, \frac{d}{L}\right)$$

③ $Re \gg 1 \rightarrow \frac{\rho g^{1/2} L^{3/2}}{\mu} \rightarrow \infty$

$$\frac{t_d}{\sqrt{L/g}} = f_1\left(\frac{P_0 - P_a}{\rho g L}, \frac{D}{L}, \frac{H}{L}, \frac{d}{L}\right) ; \quad F_x = f_2\left(\frac{P_0 - P_a}{\rho g L}, \frac{D}{L}, \frac{H}{L}, \frac{d}{L}\right)$$

④ $D_1 = 10m$, $H_1 = 5m$, $d_1 = 2cm$, $L_1 = 50m$ / $P_{01} = 2 \cdot 10^5 Pa$ → MARS.

1 → MARS
2 → EARTH

$$\left.\begin{aligned} \left(\frac{P_0 - P_a}{\rho g L}\right)_1 &= \left(\frac{P_0 - P_a}{\rho g L}\right)_2 \\ \left(\frac{D}{L}\right)_1 &= \left(\frac{D}{L}\right)_2 \\ \left(\frac{H}{L}\right)_1 &= \left(\frac{H}{L}\right)_2 \\ \left(\frac{d}{L}\right)_1 &= \left(\frac{d}{L}\right)_2 \end{aligned}\right\} \Rightarrow \left(\frac{t_d}{\sqrt{L/g}}\right)_1 = \left(\frac{t_d}{\sqrt{L/g}}\right)_2$$

$D_2 = \frac{L_2}{L_1} D_1 = \frac{10m}{50} = 0.2\,m$

$H_2 = \frac{L_2}{L_1} D_1 = \frac{5m}{50} = 0.1\,m$

$d_2 = \frac{L_2}{L_1} d_1 = \frac{2cm}{50} = 4 \cdot 10^{-4} m$

$(P_0 - P_a)_2 = \frac{(\rho g L)_2}{(\rho g L)_1} (P_0 - P_a)_1 = \frac{9.8}{3.7} \frac{1}{50} (P_0 - P_a)_1 = 10563\,Pa$

$t_{d_1} = t_{d_2} \left(\frac{g_2}{g_1}\right)^{1/2} \left(\frac{L_1}{L_2}\right)^{1/2} = 115h$

$Re = \frac{\rho g^{1/2} L^{3/2}}{\mu} \sim 3 \cdot 10^6$ (EARTH)
$1.4 \cdot 10^7$ (MARS)

BOTH $Re \gg 1$.

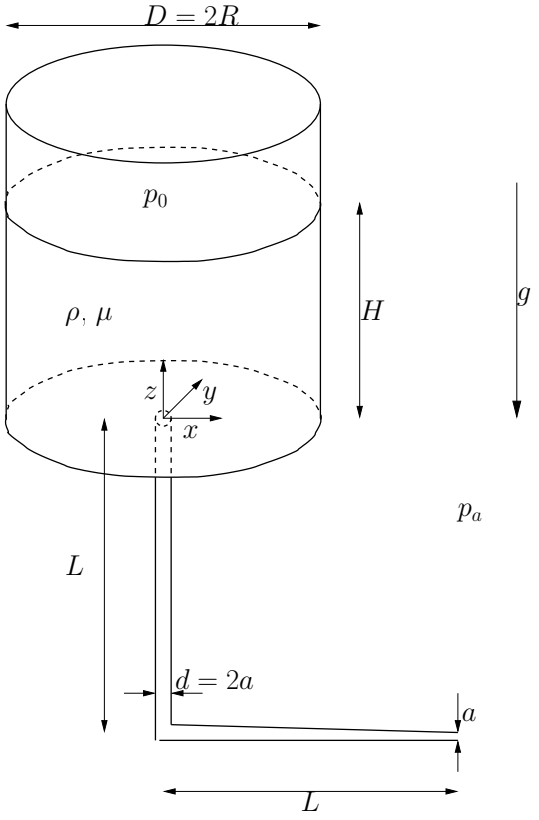
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INTRODUCTION TO FLUID MECHANICS. **17-01-2017**
PROBLEM 4 (10 POINTS). **Time: 50 minutes**

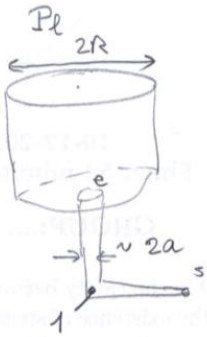
NAME:..... **GROUP:.....**

The sketch shows a cylindrical tank and al L-shaped pipe made of a straight pipe of length L and diameter $d = 2a$, a 90^o elbow and a horizontal conical pipe with a radius varying linearly with x from $d = 2a$ to $d = a$ at the exit. The tank contains, initially, a liquid of density ρ and viscosity μ, up to a level H, and a gas pressurized to a pressure p_0, that will remain constant in the discharge process to be analyzed. Starting at time $t = 0$, the liquid discharges in the open atmosphere through the exit of the pipe and the level $h(t)$ of liquid in the tank will decrease until the complete discharge of the liquid in the tank, at $t = t_{\rm d}$. In the discharge process, the fluids exert a force F_x, on the tank and on the pipe, in the x-direction. This force will be a function of t through $h(t)$.

1. Give the criteria for the flow to be dominated by viscosity in the pipe.
2. Obtain the velocity profile along the **whole** pipe
3. Obtain the reduced pressure distribution along the pipe
4. Determine the tank discharge time $t_{\rm d}$



Note: $R \gg a$, $L \gg a$, $H \sim L$, $(p_0 - p_a) \sim \rho g L$

① 

$$P_l \sim \frac{P_e - P_s}{2L} \sim \frac{P_0 + P_a + \rho g(h+L)}{2L} \sim \rho g$$

$$\rho \frac{\partial \bar{v}}{\partial t} + \rho \bar{v}\cdot\nabla\bar{v} = -\nabla P + \mu\left[\nabla_L^2 \bar{v} + \nabla_\perp^2 \bar{v}\right]$$

$$\frac{\rho V_L}{t_c} \qquad \frac{\rho V_L^2}{L} \qquad P_l \qquad \cancel{\frac{\mu V_L}{L^2}} \ (R \ll L) \qquad \frac{\mu V_L}{a^2}$$

$t_c \sim t_d \sim \frac{HR^2}{\nu d^2}$, where $\boxed{V_L \sim \frac{P_l a^2}{\mu} \sim \frac{g a^2}{\nu}} \rightarrow t_d \sim \frac{HR^2}{g a^4}\nu$

$$\frac{O\left(\rho \frac{\partial \bar{v}}{\partial t}\right)}{O\left(\mu \nabla_\perp^2 \bar{v}\right)} \sim \frac{\rho \frac{V_L}{t_c}}{\mu \frac{V_L}{a^2}} \sim \frac{a^2}{\nu t_c} \sim \frac{g a^6}{\nu^2 HR^2} \sim \frac{g a^4}{\nu^2 H}\left(\frac{a}{R}\right)^2 \ll 1.$$

$$\frac{O(\rho \bar{v}\cdot\nabla\bar{v})}{O(\mu \nabla_\perp^2 \bar{v})} \sim \frac{\rho \frac{V_L}{L}}{\mu \frac{V_L}{a^2}} \sim \frac{\rho V_L a^2}{\mu L} \sim \frac{g a^4}{\nu^2 L} \ll 1 \quad \text{ENOUGH, SINCE } \left(\frac{a}{R}\right) \ll 1$$

② VERTICAL PIPE:



$$\begin{cases} 0 = -\frac{\partial P}{\partial z} + \frac{\mu}{r}\frac{\partial}{\partial r}\left(r \frac{\partial w}{\partial r}\right) \\ \frac{\partial w}{\partial r}(r=0) = 0 \\ w(r=a) = 0 \end{cases} \rightarrow w = \frac{\partial P}{\partial z}\frac{r^2}{4\mu} + B$$

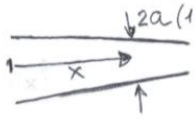
$$\boxed{w = -\frac{\partial P}{\partial z}\frac{a^2 - r^2}{4\mu}} \quad < 0 \text{ since FLOW DOWNWARDS.}$$

$$Q(t) = -\int_0^a 2\pi r\, w(r)\, dr = +P_z \frac{\pi a^4}{8\mu}; \quad \text{SINCE } Q(t) \Rightarrow P_z(t) \text{ a=const.}$$

$$P_z = \frac{P_e - P_1}{L} = \frac{P_0 + \rho g h(t) - P_1}{L}$$

$$Q(t) = \frac{P_0 + \rho g h - P_1}{L} \cdot \frac{\pi a^4}{8\mu}, \quad P_1 \text{ TO BE DETERMINED.}$$

HORIZONTAL PIPE



$$\begin{cases} 0 = -\frac{\partial P}{\partial x} + \frac{\mu}{r}\frac{\partial}{\partial r}\left(r \frac{\partial u}{\partial r}\right) \\ u(r=a) = 0 \\ \frac{\partial u}{\partial r}(r=0) = 0 \end{cases} \rightarrow u = -\frac{P_x}{4\mu}\left(a(x)^2 - r^2\right)$$

$$Q = \int_0^{a(x)} 2\pi r\, u(r)\, dr = -\frac{\pi a^4(x)}{8\mu} P_x; \quad \begin{matrix} Q = \text{const.} \\ a(x) \neq \text{const} \end{matrix} \rightarrow P_x \neq \text{const.}$$

$$-\frac{\partial P}{\partial x} = \frac{8\mu Q}{\pi a^4(x)} \rightarrow P_1 - P(x) = \frac{8\mu Q}{\pi}\int_0^x \frac{dx}{a^4(x)} = \frac{8\mu Q}{\pi a^4}\int_0^x \frac{dx}{\left(1 - \frac{x}{2L}\right)^4}$$

$$P_1 - P(x) = \frac{8\mu Q}{\pi a^4}\left(1 - \frac{x}{2L}\right)^{-3}\left(-\frac{1}{3}\right)\cdot(-2L)\Big]_0^x = \frac{16}{3}\frac{\mu Q L}{\pi a^4}\left(1 - \frac{x}{2L}\right)^{-3}$$

$$P_1 - P(x=L) = P_1 - (P_a - \rho g L) = \frac{128}{3}\frac{\mu Q L}{\pi a^4} \rightarrow P_1 = P_a - \rho g L + \frac{128}{3}\frac{\mu Q L}{\pi a^4}$$

FROM RESULT FOR VERTICAL PIPE:

$$Q = \frac{P_0 - P_a + \rho g(h+L) - \frac{128}{3}\frac{\mu Q L}{\pi a^4}}{L} \cdot \frac{\pi a^4}{8\mu}$$

$$Q = \frac{P_0 - P_a + \rho g(h+L)}{L}\frac{\pi a^4}{8\mu} - \frac{16}{3}Q \Rightarrow \boxed{Q = \frac{3\pi a^4}{128\mu}\frac{(P_0 - P_a + \rho g(h+L))}{L}}$$

DISCHARGE TIME:

$$\pi R^2 \frac{dh}{dt} = -Q$$

$$\begin{cases} \pi R^2 \frac{dh}{dt} = -\frac{3\pi a^4}{128\mu}\frac{(P_0 - P_a + \rho g(h+L))}{L} \\ h(0) = H \end{cases}$$

$$\int_H^0 \frac{dh}{P_0 - P_a + \rho g(h+L)} = \int_0^{t_d} -\frac{3}{128}\frac{a^4}{R^2 \mu L}\, dt$$

$$\frac{1}{\rho g}\ln(P_0 - P_a + \rho g(h+L))\Big]_0^H = \frac{3}{128}\frac{a^4}{R^2 \mu L} t_d$$

$$\boxed{\ln\left[\frac{P_0 - P_a + \rho g(H+L)}{P_0 - P_a + \rho g L}\right] = \frac{3}{128}\frac{a^4 \rho g}{\mu L R^2} t_d}$$