ESCUELA POLITÉCNICA SUPERIOR

INTRODUCTION TO FLUID MECHANICS. **18-06-2015**
PROBLEM 1 (10 POINTS). **Time: 60 minutes**

NAME:..... **GROUP:.....**

A thin film of liquid of density ρ and viscosity μ collides, at a velocity V_e and temperature T_e, with a cylinder of radius R. The thickness of the film is $h_e \ll R$ and the angle of incidence is φ_e. Upon impact, the jet is split in two layers, of thicknesses h_1 and h_2, that cover completely the surface of the cylinder and merge again at a point diametrally opposed to that of the impact, where a new film of is formed that separates from the cylinder surface. A known heat per unit of time $\dot{q}$ is transfered from the cylinder to the liquid.

If viscous effects, mass forces and heat transfer between the liquid and the air are negligible, the flow is steady and the surrounding air can be considered to be at constant pressure p_a:

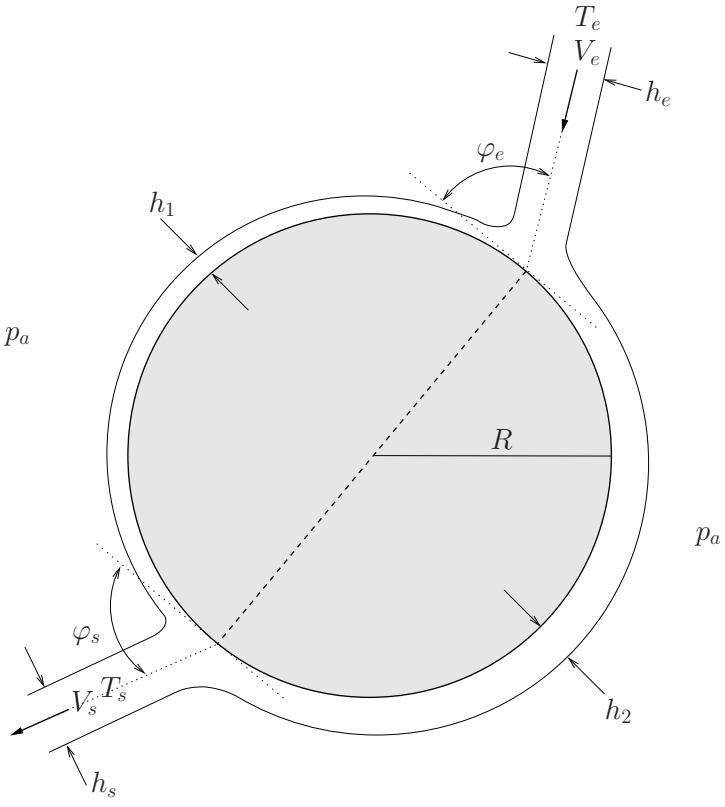
1. Use Bernoulli equation and conservation principles to determine h_1, h_2 and the velocities V_1 and V_2 of the fluid. Take into account that, since $h_e \ll R$, for this local analysis the surface of the cylinder can be substituted for the tangent plane at the impigning point (see detailed sketch).

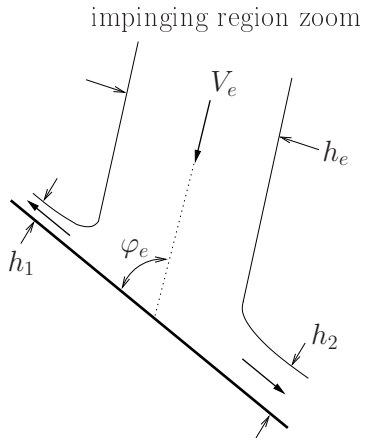
2. Do a similar analysis at the separation point, to find the angle φ_s, the thickness h_s and the velocity V_s of the detached jet.

3. Find the force exerted by the liquid on the cylinder, per unit of length.

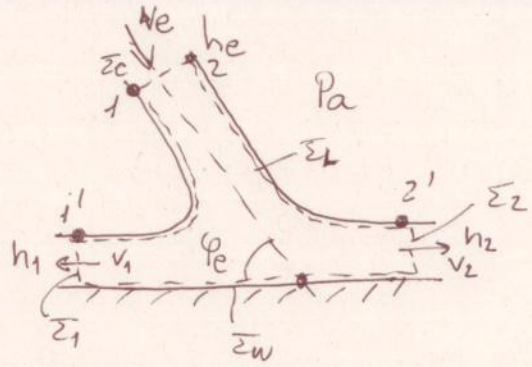
4. Obtain the temperature T_s of the liquid when the jet gets detached of the cylinder.

Known data: R, ρ, V_e, T_e, h_e, φ_e, p_a, $\dot{q}$





①



$\underline{1-1'}$: $P_a + \frac{1}{2}\rho V_e^2 = P_a + \frac{1}{2}\rho V_1^2 \Rightarrow \boxed{V_1 = V_e}$

$\underline{2-2'}$: $P_a + \frac{1}{2}\rho V_e^2 = P_a + \frac{1}{2}\rho V_2^2 \Rightarrow \boxed{V_2 = V_e}$

CONTINUITY: $\frac{d}{dt}\int_{V_c}\rho\,dV + \int_{\Sigma_c}\rho(\bar{v}-\bar{v}_c)\cdot\bar{n}\,d\sigma = 0$ (0 on Σ_1, Σ_2, Σ_L, Σ_w)

$-V_e h_e + V_e h_1 + V_e h_2 = 0$

$$\boxed{h_1 + h_2 = h_e} \quad (1a)$$

MOMENTUM: $\frac{d}{dt}\int_{V_c}\rho\bar{v}\,dV + \int_{\Sigma_c}\rho\bar{v}(\bar{v}-\bar{v}_c)\cdot\bar{n}\,d\sigma = \int_{\Sigma_c}-p\bar{n}\,d\sigma + \int_{\Sigma_c}\bar{\tau}'\cdot\bar{n}\,d\sigma + \int_{V_c}\rho\bar{f}_m\,dV$

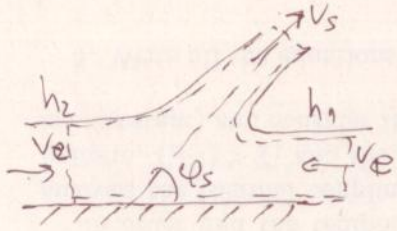
$+\int_{\Sigma_L}\bar{p}_a\bar{n}\,d\sigma$ (0) $= \bar{F}_{S\to L}$, direction $\bar{n}$

TANGENTIAL COMPONENT: $-\rho V_e^2 h_e \cos\varphi_e + \rho V_e^2 h_2 - \rho V_e^2 h_1 = 0$

$$\boxed{h_1 - h_2 = h_e \cos\varphi_e} \quad (1b)$$

(1a), (1b) $\Rightarrow$ $$\boxed{h_1 = \frac{h_e(1+\cos\varphi_e)}{2} \;;\; h_2 = \frac{h_e(1-\cos\varphi_e)}{2}}$$

②



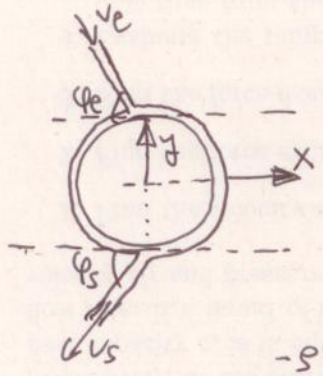
Bernoulli $\rightarrow$ $\boxed{V_s = V_e}$

continuity: $\boxed{h_s = h_1 + h_2 = h_e}$

momentum tangential direction: $h_s\rho V_s^2\cos\varphi_s - \rho V_e^2 h_2 + \rho V_e^2 h_1 = 0$

$$\cos\varphi_s = \frac{h_1 - h_2}{h_e} = \cos\varphi_e \rightarrow \boxed{\varphi_s = \varphi_e}$$

③



MOMENTUM: $\frac{d}{dt}\int_{V_c}\rho\bar{v}\,dV + \int_{\Sigma_c}\rho\bar{v}(\bar{v}-\bar{v}_c)\cdot\bar{n}\,d\sigma = \int_{\Sigma_c}-(p-p_a)\bar{n}\,d\sigma + \int_{\Sigma_c}\bar{\tau}'\cdot\bar{n}\,d\sigma + \int_{V_c}\rho\bar{f}_m\,dV$

$+\int_{\Sigma_c}p_a\bar{n}\,d\sigma$ (0) $= \bar{F}_{C\to L} = -\bar{F}_{L\to C}$

$$-\rho V_e h_e \begin{Bmatrix} V_e\cos\varphi_e \\ -V_e\sin\varphi_e \end{Bmatrix} + \rho V_e h_e \begin{Bmatrix} -V_e\cos\varphi_s \\ -V_e\sin\varphi_s \end{Bmatrix} = -\bar{F}$$

$$\boxed{\bar{F} = 2\rho V_e^2 h_e \cos\varphi_e\,\bar{e}_x}$$

④ SAME C.V. ENERGY: $\frac{d}{dt}\int_{V_c}\rho\left(e+\frac{v^2}{2}\right)dV + \int_{\Sigma_c}\rho\left(e+\frac{v^2}{2}\right)(\bar{v}-\bar{v}_c)\cdot\bar{n}\,d\sigma = \int_{\Sigma_c}-p\bar{v}\cdot\bar{n}\,d\sigma + \int_{\Sigma_c}\bar{v}\cdot\bar{\tau}'\cdot\bar{n}\,d\sigma + \int_{V_c}\rho\bar{f}_m\cdot\bar{v}\,dV$

$+\int_{\Sigma_c}-\bar{q}\cdot\bar{n}\,d\sigma + \int_{V_c}(Q_r+Q_t)\,dV$ (0 on Σ_1, Σ_2, Σ_L; $\dot{q}$ on Σ_w)

$$-\rho V_e h_e\left(cT_e + \frac{V_e^2}{2}\right) + \rho V_e h_e\left(cT_s + \frac{V_e^2}{2}\right) = p_a V_e h_e - p_a V_e h_e + \dot{q}$$

$$\boxed{T_s = T_e + \frac{\dot{q}}{\rho c V_e h_e}}$$

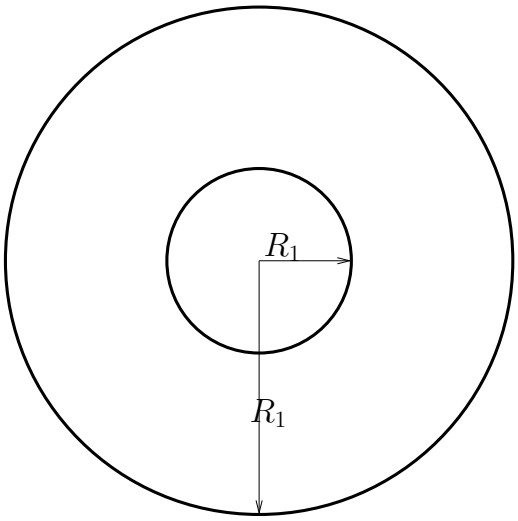
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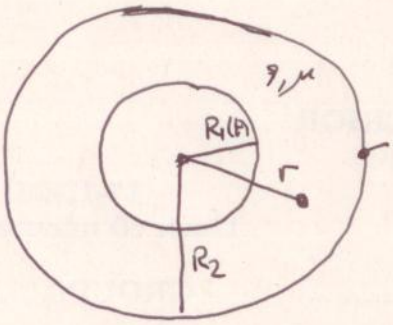
INTRODUCTION TO FLUID MECHANICS. **18-06-2015**
PROBLEM 2 (10 POINTS). **Time: 60 minutes**

NAME:..... **GROUP:.....**

In order to isolate a wire of radius R_1, we will use a layer of liquid of constant density ρ and viscosity μ, with transversal area A. This layer is externally covered by a sheath of radius $R_2 = (A/\pi + R_1^2)^{1/2}$, as sketched below. We want to study the dynamics of the movement induced in the liquid by the variations, due to thermal dilatation, of the wire radius $R_1(t)$. The elastic properties of the external sheath give us a proportional relation between pressure and R_2, $p_2 = KR_2$, where K is a constant. Assuming $R_1(t)$ is known,

1. Obtain the radial component of the velocity in the liquid $u(r,t)$.
2. Obtain the vorticity $\bar{\omega} = \nabla \wedge \bar{v}$.
3. Determine the pressure field and the viscous stress tensor τ' in the fluid.
4. Evaluate the work per unit of time exerted by the wire on the fluid.
5. Determine the viscous dissipation in the fluid field $\int \phi_v 2\pi r dr$
6. Find the temporal variation of the kinetic energy within the fluid field.
7. Find the work per unit of time exerted by the fluid on the elastic sheath.
8. Explain the relation between the results obtained in questions 4 to 7.





① Continuity: $\frac{\partial (ru)}{\partial r} = 0 \rightarrow ru = \text{const.} = R_1 \dot{R}_1 \rightarrow \boxed{u = \frac{R_1 \dot{R}_1}{r}}$

② Vorticity $\bar{\omega} = \nabla \wedge \bar{v} = 0$

③ $\rho \frac{\partial u}{\partial t} + \rho u \frac{\partial u}{\partial r} = -\frac{\partial P}{\partial r} + \mu \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial (ru)}{\partial r}\right) + 0$

$$\rho \left[\frac{\dot{R}_1^2}{r} + \frac{R_1 \ddot{R}_1}{r}\right] + \rho \frac{R_1 \dot{R}_1}{r}\left(-\frac{R_1 \dot{R}_1}{r^2}\right) = -\frac{\partial P}{\partial r}$$

$$P = -\rho (\dot{R}_1^2 + R_1 \ddot{R}_1) \ln r - \frac{1}{2} \rho \frac{R_1^2 \dot{R}_1^2}{r^2} + A$$

$$P(R_2) = k R_2 = -\rho (\dot{R}_1^2 + R_1 \ddot{R}_1) \ln R_2 - \frac{1}{2} \rho \frac{R_1^2 \dot{R}_1^2}{R_2^2} + A$$

$$\boxed{P = k R_2 + \rho (\dot{R}_1^2 + R_1 \ddot{R}_1) \ln\left(\frac{R_2}{r}\right) + \frac{\rho}{2} R_1^2 \dot{R}_1^2 \left(\frac{1}{R_2^2} - \frac{1}{r^2}\right)}$$

$$\tau'_{rr} = 2\mu \frac{\partial u}{\partial r} = -2\mu \frac{R_1 \dot{R}_1}{r^2} \; ; \quad \tau'_{\theta\theta} = 2\mu \frac{u}{r} = \frac{2\mu R_1 \dot{R}_1}{r^2} \; ;$$

$$\tau'_{r\theta} = \tau'_{rz} = \tau'_{\theta z} = \tau'_{zz} = 0$$

④ $\bar{n}$

$$\dot{W}_1 = \int_{\Sigma_{R_1}} -P \bar{v} \cdot \bar{n} \, d\sigma + \int_{\Sigma_{R_1}} \bar{v} \cdot (\bar{\tau}' \cdot \bar{n}) \, d\sigma$$

$\bar{n} = (-1, 0, 0)$; $\bar{v} = \dot{R}_1 \bar{e}_r$;

$$\int_{\Sigma_{R_1}} -P \bar{v} \cdot \bar{n} \, d\sigma = 2\pi R_1 \cdot \dot{R}_1 \left[k R_2 + \rho (\dot{R}_1^2 + R_1 \ddot{R}_1) \ln \frac{R_2}{R_1} + \frac{\rho}{2} R_1^2 \dot{R}_1^2 \left(\frac{1}{R_2^2} - \frac{1}{R_1^2}\right)\right]$$

$$\int_{\Sigma_{R_1}} (\bar{v} \cdot \bar{\tau}' \cdot \bar{n}) \, d\sigma = 2\pi R_1 \cdot \frac{R_1 \dot{R}_1}{R_1} \cdot \frac{2\mu R_1 \dot{R}_1}{R_1^2} = 4\pi\mu \dot{R}_1^2$$

$$\boxed{\dot{W}_1 = 2\pi k R_1 R_2 \dot{R}_1 + 2\pi \rho R_1 \dot{R}_1 (\dot{R}_1^2 + R_1 \ddot{R}_1) \ln \frac{R_2}{R_1} + \pi \rho R_1^3 \dot{R}_1^3 \left(\frac{1}{R_2^2} - \frac{1}{R_1^2}\right) + 4\pi\mu \dot{R}_1^2}$$

⑤ $\Phi_V = \mu \left[2\left(\frac{\partial u}{\partial r}\right)^2 + 2\left(\frac{u}{r}\right)^2\right] = 2\mu \frac{R_1^2 \dot{R}_1^2}{r^4} + 2\mu \frac{R_1^2 \dot{R}_1^2}{r^4} = \frac{4\mu R_1^2 \dot{R}_1^2}{r^4}$

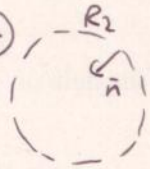
$$\boxed{\int_{R_1}^{R_2} \Phi_V \cdot 2\pi r \, dr = 4\pi\mu R_1^2 \dot{R}_1^2 \left(\frac{1}{R_1^2} - \frac{1}{R_2^2}\right)}$$

⑥ $\bar{E}_c = \int_{R_1}^{R_2} \rho \frac{v^2}{2} 2\pi r\, dr = \int_{R_1}^{R_2} \frac{1}{2} \rho \cdot 2\pi \frac{R_1^2 \dot{R}_1^2}{r^2} \cdot r\, dr = \pi \rho R_1^2 \dot{R}_1^2 \ln\left(\frac{R_2}{R_1}\right)$

$\frac{d\bar{E}_c}{dt} = 2\pi\rho R_1 \dot{R}_1 (\dot{R}_1^2 + R_1 \ddot{R}_1) \ln\frac{R_2}{R_1} + \pi\rho R_1^2 \dot{R}_1^2 \frac{\dot{R}_2 R_1 - R_2 \dot{R}_1}{R_1^2} \cdot \frac{R_1}{R_2}$

since $\pi(R_2^2 - R_1^2) = \text{const.} \rightarrow R_2\dot{R}_2 = R_1\dot{R}_1 \rightarrow \dot{R}_2 = \frac{R_1}{R_2}\dot{R}_1$

so: $\boxed{\frac{d\bar{E}_c}{dt} = 2\pi\rho R_1 \dot{R}_1 (\dot{R}_1^2 + R_1\ddot{R}_1) \ln\frac{R_2}{R_1} + \pi\rho (R_1\dot{R}_1)^3 \left(\frac{1}{R_2^2} - \frac{1}{R_1^2}\right)}$

⑦ 

$\dot{W}_2 = \int_{\Sigma_{R_2}} -p\,\bar{v}\cdot\bar{n}\,d\sigma + \int_{\Sigma_{R_2}} \bar{v}\cdot\bar{\bar{\tau}}'\cdot\bar{n}\,d\sigma = K R_2 \cdot 2\pi R_2 \cdot \frac{R_1\dot{R}_1}{R_2} + 2\pi R_2 \cdot 2\mu \frac{R_1\dot{R}_1}{R_2^2} \cdot \frac{R_1\dot{R}_1}{R_2}$

$\bar{v} = \dot{R}_2 \bar{e}_r$

$\bar{n} = (-1,0,0)$

$\boxed{\dot{W}_2 = 2\pi K R_2 R_1\dot{R}_1 + 4\pi\mu \frac{R_1^2\dot{R}_1^2}{R_2^2}}$

⑧ kinetic energy equation:

$\frac{d}{dt}\int_{V_c} \rho\frac{v^2}{2} dV + \int_{\Sigma_c} \rho\frac{v^2}{2}(\bar{v}-\bar{u})\cdot\bar{n}\,d\sigma = \int_{\Sigma} -p\bar{v}\cdot\bar{n}\,d\sigma + \int_{\Sigma} \bar{v}\cdot\bar{\bar{\tau}}'\cdot\bar{n}\,d\sigma + \int_{V_c} \rho\bar{f}_m\cdot\bar{v}\,dV$

($\int_{\Sigma_c}$: $\bar{u} = 0$; $\int_{V_c} \rho\bar{f}_m\cdot\bar{v}\,dV$: $\bar{f}_m = 0$)

$- \left[\int_{V_c} -p\,\nabla\cdot\bar{v}\,dV + \int_{V_c} \phi_v\,dV\right]$

($\int_{V_c} -p\,\nabla\cdot\bar{v}\,dV$: $\nabla\cdot\bar{v} = 0$)



$\frac{dE_c}{dt} = \dot{W}_1 - \dot{W}_2 - \int_{R_1}^{R_2} \phi_v \cdot 2\pi r\, dr$

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INTRODUCTION TO FLUID MECHANICS. **18-06-2015**
PROBLEM 3 (10 POINTS). **Time: 60 minutes**

NAME:..... **GROUP:.....**

A spherical droplet of a liquid with density ρ, viscositu μ and radius R is floating at rest, without gravity, in air of density ρ_a and viscosity μ_a. If the surface tension σ has dimensions of Force per unit length (Force/Length):

1. Use Dimensional Analysis to reduce the parametric dependence of the oscillation period T.
2. Simplify the above result if the influence of the surrounding air is negligible because $\rho_a \ll \rho$ and $\mu_a \ll \mu$.
3. Obtain an explicit expression for T (except for a constant), valid in the limit of small viscous forces in the oscillatory movement of the droplet. Give the criterion for this result to be valid.
4. In the same limit as in the previous question, obtain the ratio between the oscillation periods for two droplets of radii R and $2R$ respectively.

① $T = T(\rho, \mu, \rho_a, \mu_a, R, \sigma)$

	ρ	μ	ρ_a	μ_a	R	σ	T
M	1	1	1	1	0	1	0
L	-3	-1	-3	-1	1	0	0
T	0	-1	0	-1	0	-2	1

$$\begin{vmatrix} 1 & 0 & 1 \\ -3 & 1 & 0 \\ 0 & 0 & -2 \end{vmatrix} \neq 0 \Rightarrow \rho, R, \sigma \quad \text{D.I} \longrightarrow \text{Reduce 6 parameters - 3 parameters}$$

$$\pi_T = \frac{T \cdot \sigma^{1/2}}{\rho^{1/2} R^{3/2}} \; ; \quad \pi_\mu = \frac{\mu}{(\rho R \sigma)^{1/2}} \; ; \quad \pi_{\rho_a} = \frac{\rho_a}{\rho} \; ; \quad \pi_{\mu_a} = \frac{\mu_a}{\mu} \cdot \frac{(\rho R \sigma)^{1/2}}{\mu^{-1}}$$

$$\frac{T \sigma^{1/2}}{\rho^{1/2} R^{3/2}} = f\left(\frac{\mu}{(\rho R \sigma)^{1/2}}, \frac{\rho_a}{\rho}, \frac{\mu_a}{\mu} \cdot \frac{\mu}{(\rho R \sigma)^{1/2}}\right)$$

② $\frac{\rho_a}{\rho} \ll 1$; $\frac{\mu_a}{\mu} \ll 1$ $\Rightarrow$ $\boxed{\frac{T \sigma^{1/2}}{\rho^{1/2} R^{3/2}} = f\left(\frac{\mu}{(\rho R \sigma)^{1/2}}\right)}$

③ VISCOUS FORCES NEGLIGIBLE IF $\frac{\mu}{(\rho R \sigma)^{1/2}} \ll 1$.

$$\boxed{\frac{T \sigma^{1/2}}{\rho^{1/2} R^{3/2}} = C}$$

④ $$\frac{T_R \sigma^{1/2}}{\rho^{1/2} R^{3/2}} = \frac{T_{2R} \cdot \sigma^{1/2}}{\rho^{1/2} (2R)^{3/2}} \Rightarrow \boxed{T_{2R} = 2\sqrt{2}\, T_R}$$

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INTRODUCTION TO FLUID MECHANICS. **18-06-2015**
PROBLEM 4 (10 POINTS). **Time: 60 minutes**

NAME:..... **GROUP:.....**

We wish to find the terminal velocity U of a two-dimensional body of mass M falling, under the action of the gravity, inside a channel of width $2(d+h)$. The channel contains a liquid of density ρ and viscosity μ and it is closed in the lower end, so that we can onsider the liquid to be at rest far from the falling body. If the channel is very long, at a certain moment during the free fall of the body shown in the sketch, there is no effect of the ends of the channel and we can consider the channel as infinitely long.

1. Perform the change from a laboratory based reference system to a reference system attached to the falling body. Do a sketch showing the most important features of the flow. In the following, it is suggested to use this reference system for the analysis of the flow around the body.

2. Show that, far upstream and downstream from the body, the reduced pressure takes constant values P_0 and P_1 respectively.

3. Estimate the characteristic value U_c of the velocity between the falling body and the channel walls and give the criterion for the flow between the body and the walls of the channel to be dominated by viscosity. From this moment, we will consider the flow to be dominated by viscosity.

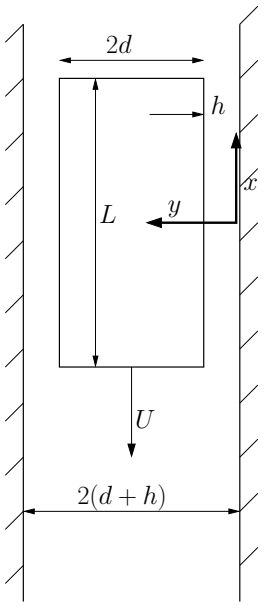
4. Estimate the pressure variation $\Delta_E P$ that takes place in the entrance region and estimate the pressure variation $\Delta_L P$ along the body. Show that $\Delta_E P \ll \Delta_L P$.

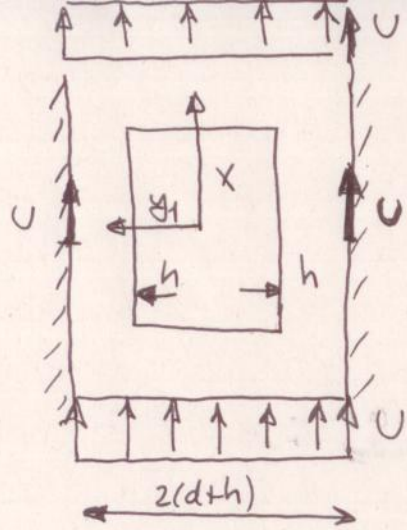
5. Use order of magnitude estimates to show that the force on the body due to viscosity F_μ and the force due to pressure F_P, are of the same order $F_\mu/F_P \sim 1$.

6. Determine the pressure distribution P along the body, F_P and F_μ.

7. Obtain the value of the terminal velocity U.

Note $L \gg d \sim h$



① 

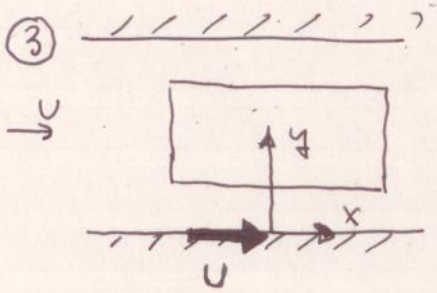
STEADY FLOW FOR $t \to \infty$.

② UPSTREAM: UNIFORM VELOCITY.

$$\left.\begin{matrix} u = U \\ v = 0 \end{matrix}\right\} \to \frac{\partial P}{\partial x} = \frac{\partial P}{\partial y} = 0 \Rightarrow P = \text{const} = P_0$$

DOWNSTREAM: UNIFORM VELOCITY

$$\left.\begin{matrix} u = U \\ v = 0 \end{matrix}\right\} \quad P = \text{const.} = P_1$$

③ 

Continuity: $U \cdot 2(d+h) \sim 2 U_c \cdot h$

$$U_c \sim U\frac{(d+h)}{h} \sim U; \quad (d \sim h.)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\frac{U}{L} \sim \frac{V}{h}$$

FLOW DOMINATED BY VISCOSITY: $\dfrac{\frac{\rho U}{t_0}}{\frac{\mu U}{h^2}} = \dfrac{\rho h^2}{\mu t_0} \ll 1$

$$\frac{\frac{\rho U^2}{L}}{\frac{\mu U}{h^2}} = \frac{\rho U h}{\mu} \cdot \frac{h}{L} \ll 1.$$

$$\rho \frac{\partial u}{\partial t} + \rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = -\frac{\partial P}{\partial x} + \mu \frac{\partial^2 u}{\partial x^2} + \mu \frac{\partial^2 u}{\partial y^2}$$

$\frac{\rho U}{t_0}$ $\quad$ $\frac{\rho U^2}{L}$ $\quad$ $\frac{\Delta_L P}{L}$ $\quad$ $\frac{\mu U}{L^2} \ll \frac{\mu U}{h^2}$

④ $\Delta_E P \sim \begin{cases} \frac{\mu U}{h}, & \text{if } Re = \frac{\rho U h}{\mu} \ll 1 \\ \rho U^2, & \text{if } Re \gg 1 \end{cases}$

$$\frac{\Delta_E P}{\Delta_L P} \sim \frac{h}{L} \ll 1.$$

$$\frac{\Delta_E P}{\Delta_L P} \sim \frac{\rho U h}{\mu} \cdot \frac{h}{L} \ll 1.$$

$$\Delta_L P \sim \frac{\mu U}{h^2} L$$

⑤ $F_P \sim \Delta_L P \cdot d \sim \frac{\mu U L}{h}$

$$F_\mu \sim \mu \cdot \frac{U}{h} \cdot L$$

⑥ $0 = -\frac{\partial P}{\partial x} + \mu \frac{\partial^2 u}{\partial y^2}$

$y=0: u=U$

$y=h: u=0$

$$u = \frac{1}{2\mu}\cdot\frac{\partial P}{\partial x}\cdot y^2 - \frac{1}{2\mu}\frac{\partial P}{\partial x} h y + U\left(1-\frac{y}{h}\right)$$

$$\frac{Q}{2} = \int_0^h u\,dy = -\frac{1}{12\mu}\frac{\partial P}{\partial x} h^3 + \frac{Uh}{2}$$

So: $\frac{\partial P}{\partial x} = \frac{6\mu}{h^3}\left[Uh - Q\right]$; $Q = 2U(d+h)$

$= -\frac{6\mu U}{h^3}\left[2d + h\right] = -\frac{6\mu U}{h^2}\left[\frac{2d}{h}+1\right]$ (notice $\Delta_L P \sim \frac{\mu U L}{h^2}$)

$$P(x) - P_0 = -\frac{6\mu U x}{h^2}\left[\frac{2d}{h}+1\right] \; ; \quad \boxed{P_0 - P_1 = \frac{6\mu U L}{h^2}\left[\frac{2d}{h}+1\right]}$$

if $\Delta_L P$ as small: $F_P = (P_0 - P_1).2d = \frac{12\mu U L d}{h^2}\left[\frac{2d}{h}+1\right]$

Viscous stress at the wall of the body:

$$\mu \left.\frac{\partial u}{\partial y}\right|_{y=h} = \mu \frac{\partial u}{\partial y} = \left[\frac{y}{2}\frac{\partial P}{\partial x} - \frac{h}{2}\frac{\partial P}{\partial x} - \frac{\mu U}{h}\right]_{y=h} = \frac{h}{2}\frac{\partial P}{\partial x} - \frac{\mu U}{h}$$

$$= -\frac{6\mu U h}{h^3}\frac{h}{2}\left[2d+h\right] - \frac{\mu U}{h} = -\frac{\mu U}{h}\left[6\frac{d}{h} + 4\right]$$

$$F_\mu = -\mu \left.\frac{\partial u}{\partial y}\right|_{y=h} \cdot L \cdot 2 = \frac{4\mu U L}{h}\left[3\frac{d}{h} + 2\right]$$

⑦ Balance : $F_P + F_\mu = Mg$

$$\frac{12\mu U L}{h}\left[3\left(\frac{d}{h}\right)^2 + 3\frac{d}{h} + 1\right] = Mg$$

$$\boxed{U = \frac{Mgh}{12\mu L\left[3\left(\frac{d}{h}\right)^2 + 3\frac{d}{h} + 1\right]}}$$