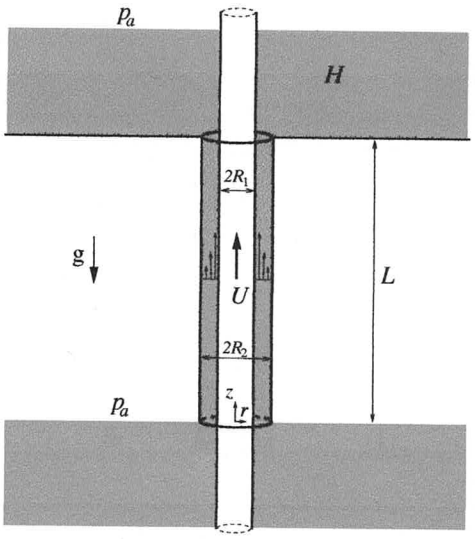
Problem 4:

To fill an open container of depth H located at a height L above the surface of a reservoir we use the viscous pumping system sketched in the figure, including a pipe or radius R_2 connecting the surface of the reservoir with the bottom of the container and an infinitely long, inner coaxial rod of radius R_1 that moves vertically with velocity U, dragging the water into the inter-cylinder gap.

1. Give the condition for the flow in the gap to be dominated by viscosity.
2. Obtain the velocity profile $v_z(r)$ across the gap.
3. Determine the loss of reduced pressure per unit length $P_l = -\partial P/\partial z$.
4. Compute the force acting on the moving shaft and the power needed to move it.
5. Determine the volume flow rate Q of the pumping system, including the minimum velocity U_{min} required for $Q > 0$.



1) $\frac{\rho U (R_2-R_1)^2}{\mu L} \ll 1$ 2) $0 = P_\ell + \frac{\mu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)$ $\begin{cases} r=R_1: v_z = U \\ r=R_2: v_z = 0 \end{cases}$ $\Rightarrow$ $v_z = \frac{P_\ell}{4\mu}(R_2^2 - r^2) + \left[U - \frac{P_\ell}{4\mu}(R_2^2 - R_1^2)\right]\frac{\ln(R_2/r)}{\ln(R_2/R_1)}$

3) $P_\ell = \text{constant} = -\frac{\partial P}{\partial z} = \frac{P_a - [P_a + \rho g(H+L)]}{L} = -\frac{\rho g (H+L)}{L}$

4) $W = U\, 2\pi R_1 L\, \tau_w$ (Force) $\quad \tau_w = \mu \left.\frac{\partial v_z}{\partial r}\right|_{r=R_1} = -\frac{P_\ell}{2}R_1 - \frac{\mu}{R_1}\frac{\left[U - \frac{P_\ell}{4\mu}(R_2^2 - R_1^2)\right]}{\ln(R_2/R_1)} =$

$= -\frac{\mu U}{R_1 \ln(R_2/R_1)} - \frac{P_\ell}{2}R_1\left[1 - \frac{1}{2}\frac{(R_2^2/R_1^2 - 1)}{\ln(R_2/R_1)}\right]$

5) $Q = \int_{R_1}^{R_2} 2\pi r\, v_z\, dr =$

$= 2\pi\left\{\frac{P_\ell}{16\mu}(R_2^4 - R_1^4) + \frac{\left(U - \frac{P_\ell}{4\mu}(R_2^2 - R_1^2)\right)}{4\ln(R_2/R_1)}\left[R_2^2(2\ln R_2 - 1) - R_1^2(2\ln R_1 - 1)\right]\right\}$

$U_{min} = -\frac{P_\ell}{4\mu}(R_2^2 - R_1^2)\left[\frac{(R_2^2 + R_1^2)\ln(R_2/R_1)}{R_2^2(2\ln R_2 - 1) - R_1^2(2\ln R_1 - 1)} - 1\right]$